

# The algebra of integro-differential operators on an affine line and its modules

V. V. Bavula

## Abstract

For the algebra  $\mathbb{I}_1 = K\langle x, \frac{d}{dx}, \int \rangle$  of polynomial integro-differential operators over a field  $K$  of characteristic zero, a classification of simple modules is given. It is proved that  $\mathbb{I}_1$  is a left and right coherent algebra. The *Strong Compact-Fredholm Alternative* is proved for  $\mathbb{I}_1$ . The endomorphism algebra of each simple  $\mathbb{I}_1$ -module is a *finite dimensional* skew field. In contrast to the first Weyl algebra, the centralizer of a non-scalar integro-differential operator can be a noncommutative, non-Noetherian, non-finitely generated algebra which is not a domain. It is proved that neither left nor right quotient ring of  $\mathbb{I}_1$  exists but there exists the *largest left quotient ring* and the *largest right quotient ring* of  $\mathbb{I}_1$ , they are not  $\mathbb{I}_1$ -isomorphic but  $\mathbb{I}_1$ -*anti-isomorphic*. Moreover, the factor ring of the largest right quotient ring modulo its only proper ideal is isomorphic to the quotient ring of the first Weyl algebra. An analogue of the Theorem of Stafford (for the Weyl algebras) is proved for  $\mathbb{I}_1$ : each *finitely generated* one-sided ideal of  $\mathbb{I}_1$  is 2-generated.

*Key Words:* the algebra of polynomial integro-differential operators, the Strong Compact-Fredholm Alternative, coherent algebra, the Weyl algebra, simple module, compact integro-differential operators, Fredholm operators, centralizer, the largest left/right quotient ring.

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## 1 Introduction

Throughout, ring means an associative ring with 1; module means a left module;  $\mathbb{N} := \{0, 1, \dots\}$  is the set of natural numbers;  $K$  is a field of characteristic zero and  $K^*$  is its group of units;  $P_1 := K[x]$  is a polynomial algebra in one variable  $x$  over  $K$ ;  $\partial := \frac{d}{dx}$ ;  $\text{End}_K(P_1)$  is the algebra of all  $K$ -linear maps from  $P_1$  to  $P_1$ , and  $\text{Aut}_K(P_1)$  is its group of units (i.e., the group of all the invertible linear maps from  $P_1$  to  $P_1$ ); the subalgebras  $A_1 := K\langle x, \partial \rangle$  and  $\mathbb{I}_1 := K\langle x, \partial, \int \rangle$  of  $\text{End}_K(P_1)$  are called the (first) *Weyl algebra* and the *algebra of polynomial integro-differential operators* respectively where  $\int : P_1 \rightarrow P_1$ ,  $p \mapsto \int p dx$ , is *integration*, i.e.,  $\int : x^n \mapsto \frac{x^{n+1}}{n+1}$  for all  $n \in \mathbb{N}$ . The algebra  $\mathbb{I}_1$  is neither left nor right Noetherian and not a domain. Moreover, it contains infinite direct sums of nonzero left and right ideals, [12].

In Section 2, a classification of simple  $\mathbb{I}_1$ -modules is given (Theorem 2.1), it is similar to the classification of simple  $A_1$ -modules from [4] and to Block's classification [17], [18] (Block was the first who did it for  $A_1$  in his seminal, long paper [18]. Later advancing Block's ideas and using an approach of generalized Weyl algebras an alternative classification was given in [4] and the proof comprises only several pages). One might expect (which is not obvious from the outset) a close connection between the simple modules of the algebras  $A_1$  and  $\mathbb{I}_1$ . The reality is even more surprising: the algebra  $\mathbb{I}_1$  has exactly 'one less' simple module than the algebra  $A_1$  (Theorem 2.2). So, surprisingly, inverting on the right the derivation  $\partial$  (i.e., adding the integration  $\int$  which is a *one-sided* inverse of  $\partial$ ,  $\partial \int = 1$ , but not two-sided) 'kills' only a *single* simple  $A_1$ -module.

In Section 3, for modules and algebras two new concepts are introduced: the Compact-Fredholm Alternative and the Strong Compact-Fredholm Alternative. The Strong Compact-Fredholm Alternative is proved for the algebra  $\mathbb{I}_1$  (Theorem 3.1) which says that on *all* simple

$\mathbb{I}_1$ -modules the action of each element of  $\mathbb{I}_1$  is either *compact* (i.e., the image is finite dimensional) or, otherwise, *Fredholm* (i.e., the kernel and the cokernel are finite dimensional). So, the algebra

$$\mathbb{I}_1 = \mathcal{C}_{\mathbb{I}_1} \coprod \mathcal{F}_{\mathbb{I}_1}$$

is the disjoint union of the sets of compact and Fredholm operators respectively. The set of compact operators  $\mathcal{C}_{\mathbb{I}_1}$  coincides with the only proper (two-sided) ideal

$$F = \bigoplus_{i,j \in \mathbb{N}} K(\int^i \partial^j - \int^{i+1} \partial^{j+1})$$

of the algebra  $\mathbb{I}_1$  (Theorem 3.1). The endomorphism algebra of each simple  $\mathbb{I}_1$ -module is a finite dimensional division algebra (Theorem 3.4).

- (Theorem 3.6) *Let  $a \in \mathbb{I}_1$ ,  $\cdot a : \mathbb{I}_1 \rightarrow \mathbb{I}_1$ ,  $b \mapsto ba$ , and  $l_{\mathbb{I}_1}$  be the length function on the set of left  $\mathbb{I}_1$ -modules. Then*

1.  $l_{\mathbb{I}_1}(\ker(\cdot a)) < \infty$  iff  $l_{\mathbb{I}_1}(\text{coker}(\cdot a)) < \infty$  iff  $a \notin F$ .
2.  $l_{\mathbb{I}_1}(\text{im}(\cdot a)) < \infty$  iff  $a \in F$ .

Various indices are introduced for non-compact integro-differential operators, the *M-index* and the *length index* and it is proved that they are invariant under addition of compact operator (Lemma 3.5 and Lemma 3.7).

In Section 4, it is proved that the algebra  $\mathbb{I}_1$  is a left and right coherent algebra (Theorem 4.4). In particular, it is proved that the intersection of finitely many finitely generated left (or right) ideals of  $\mathbb{I}_1$  is again a finitely generated left (or right) ideal of  $\mathbb{I}_1$  (Theorem 4.3).

- (Theorem 4.2) *Let  $a \in \mathbb{I}_1$ . Then*
1.  $\ker_{\mathbb{I}_1}(\cdot a)$  and  $\text{coker}_{\mathbb{I}_1}(\cdot a)$  are finitely generated left  $\mathbb{I}_1$ -modules.
  2.  $\ker_{\mathbb{I}_1}(a \cdot)$  and  $\text{coker}_{\mathbb{I}_1}(a \cdot)$  are finitely generated right  $\mathbb{I}_1$ -modules.

The Theorem of Stafford says that *each left or right ideal of the Weyl algebra  $A_n$  over a field of characteristic zero is generated by two elements* [28]. The same result holds for a larger class of algebras that includes the, so-called, generalized Weyl algebras [11]. The simplicity of the Weyl algebra  $A_n$  plays a crucial role in the proof. An analogue of this result is proved for the (non-Noetherian, non-simple) algebra  $\mathbb{I}_1$ .

- (Theorem 4.5) *Each finitely generated left (or right) ideal of the algebra  $\mathbb{I}_1$  is generated by two elements.*

Section 5: The centralizer of each non-scalar element of the Weyl algebra  $A_1$  is a finitely generated commutative (hence Noetherian) domain (see Burchall and Chaundy [19] and Amitsur [1]). In contrast to the Weyl algebra  $A_1$ , the centralizer of a non-scalar element of the algebra  $\mathbb{I}_1$  can be a non-finitely generated, noncommutative, non-Noetherian algebra which is not a domain (Proposition 5.1.(1)). Theorem 5.7 describes in great detail the structure of centralizers of non-scalar elements of the algebra  $\mathbb{I}_1$ . Theorem 5.7 is too technical to be explained in the Introduction, it is a generalization of the Theorem of Amitsur on the centralizer for the first Weyl algebra  $A_1$ . We state here only two corollaries of Theorem 5.7.

- (Corollary 5.8) *Let  $a \in \mathbb{I}_1 \setminus K$  and  $H := \partial x$ . Then the following statements are equivalent.*
1.  $a \notin K[H] + F$ .
  2.  $\text{Cen}_{\mathbb{I}_1}(a)$  is a finitely generated  $K[a]$ -module.
  3.  $\text{Cen}_{\mathbb{I}_1}(a)$  is a left Noetherian algebra.

4.  $\text{Cen}_{\mathbb{I}_1}(a)$  is a right Noetherian algebra.
  5.  $\text{Cen}_{\mathbb{I}_1}(a)$  is a finitely generated and Noetherian algebra.
- (Corollary 5.9) Let  $a \in \mathbb{I}_1 \setminus K$ . Then
    1.  $\text{Cen}_{\mathbb{I}_1}(a)$  is a finitely generated algebra iff  $a \notin (K[H] + F) \setminus (K + F)$ .
    2.  $\text{Cen}_{\mathbb{I}_1}(a)$  is a finitely generated, not Noetherian/not left Noetherian/not right Noetherian algebra iff  $a \in (K + F) \setminus K$ .
    3. The algebra  $\text{Cen}_{\mathbb{I}_1}(a)$  is not finitely generated, not Noetherian/not left Noetherian/not right Noetherian iff  $a \in (K[H] + F) \setminus (K + F)$ .

In Section 6, for all elements  $a \in \mathbb{I}_1 \setminus F$ , an explicit formula for the index  $\text{ind}(a_{K[x]})$  is found (Proposition 6.1.(1)) where  $a_{K[x]} : K[x] \rightarrow K[x]$ ,  $p \mapsto ap$ . Classifications are given of elements  $a \in \mathbb{I}_1$  that satisfy the following properties: the map  $a_{K[x]}$  is a bijection (Theorem 6.2), a surjection (Theorem 6.3), an injection (Theorem 6.6). In case when the map  $a_{K[x]}$  is a bijection an explicit inversion formula is found (Theorem 6.2.(4)). The kernel and the cokernel of the linear map  $a_{K[x]}$  are found in the cases when the map  $a_{K[x]}$  is either surjective or injective (Theorem 6.3 and Theorem 6.6).

- (Proposition 6.11)
  1. For each element  $a \in \mathbb{I}_1 \setminus F$  with  $n := \dim_K(\text{coker}(a_{K[x]}))$ , there exists an element  $\partial^n + f$  for some  $f \in F$  (resp.  $s \in (1+F)^*$ ) such that the map  $(\partial^n + f)a_{K[x]}$  (resp.  $s\partial^n s^{-1}a_{K[x]}$ ) is a surjection. In this case,  $\ker((\partial^n + f)a_{K[x]}) = \ker(a_{K[x]})$  (resp.  $\ker(s\partial^n s^{-1}a_{K[x]}) = \ker(a_{K[x]})$ ).
  2. For each element  $a \in \mathbb{I}_1 \setminus F$  with  $n := \dim_K(\ker(a_{K[x]}))$ , there exists an element  $\int^n + g$  for some  $g \in F$  (resp.  $s \in (1+F)^*$ ) such that the map  $a(\int^n + g)_{K[x]}$  (resp.  $s \int^n s^{-1}a_{K[x]}$ ) is an injection. In this case,  $\text{im}(a(\int^n + g)_{K[x]}) = \text{im}(a_{K[x]})$  (resp.  $\text{im}(s \int^n s^{-1}a_{K[x]}) = \text{im}(a_{K[x]})$ ).
  3. For each element  $a \in \mathbb{I}_1 \setminus F$  with  $m := \dim_K(\ker(a_{K[x]}))$  and  $n := \dim_K(\text{coker}(a_{K[x]}))$ , there exist elements  $\partial^n + f$  and  $\int^m + g$  for some  $f, g \in F$  (resp.  $s, t \in (1+F)^*$ ) such that the map  $(\partial^n + f)a(\int^m + g)_{K[x]}$  (resp.  $s\partial^n s^{-1}at \int^m t_{K[x]}^{-1}$ ) is a bijection.

Proposition 6.12 gives necessary and sufficient conditions for each of the vector spaces  $\ker_{\ker_{\mathbb{I}_1}(\cdot b)}(a \cdot)$ ,  $\text{coker}_{\ker_{\mathbb{I}_1}(\cdot b)}(a \cdot)$ ,  $\ker_{\text{coker}_{\mathbb{I}_1}(\cdot b)}(a \cdot)$  and  $\text{coker}_{\text{coker}_{\mathbb{I}_1}(\cdot b)}(a \cdot)$  to be finite dimensional.

The algebra  $K + F = K + M_\infty(K)$  admits the (usual) determinant map  $\det$  (see (33)), and the group of units  $\mathbb{I}_1^*$  of the algebra  $\mathbb{I}_1$  is equal to  $\{u \in K + F \mid \det(u) \neq 0\}$ , [12].

For an element  $u \in \mathbb{I}_1$ , let  $\text{l.inv}(u) := \{v \in \mathbb{I}_1 \mid vu = 1\}$  and  $\text{r.inv}(u) := \{v \in \mathbb{I}_1 \mid uv = 1\}$ , the sets of left and right inverses for the element  $u$ . In 1942, Baer [3] and, in 1950, Jacobson [21] began to study one sided inverses. The next theorem describes all the left and right inverses of elements in  $\mathbb{I}_1$ .

- (Corollary 7.2)
  1. An element  $a \in \mathbb{I}_1$  admits a left inverse iff  $a = a' \int^n$  for some natural number  $n \geq 0$  and an element  $a' \in K^* + F$  such that  $a_{K[x]}$  is an injection (necessarily,  $n = \dim_K(\text{coker}(a_{K[x]}))$ ). In this case,  $\text{l.inv}(a) = \{b \in \partial^n b' \mid b' \in K^* + F, \int^n \partial^n b' a' \int^n \partial^n = \int^n \partial^n\}$ .  $\mathcal{L}(\mathbb{I}_1) = \{a \in (K^* + F) \int^n \mid n \in \mathbb{N}, a_{K[x]} \text{ is an injection}\}$ .
  2. An element  $b \in \mathbb{I}_1$  admits a right inverse iff  $b = \partial^n b'$  for some natural number  $n \geq 0$  and an element  $b' \in K^* + F$  such that  $b_{K[x]}$  is a surjection (necessarily,  $n = \dim_K(\ker(b_{K[x]}))$ ). In this case,  $\text{r.inv}(b) = \{a' \int^n \mid a' \in K^* + F, \int^n \partial^n b' a' \int^n \partial^n = \int^n \partial^n\}$ .  $\mathcal{R}(\mathbb{I}_1) = \{b \in \partial^n (K^* + F) \mid n \in \mathbb{N}, b_{K[x]} \text{ is a surjection}\}$ .

In Section 7, it is proved that the monoid  $\mathcal{L}(\mathbb{I}_1) := \{a \in \mathbb{I}_1 \mid ba = 1 \text{ for some } b \in \mathbb{I}_1\}$  of left invertible elements of the algebra  $\mathbb{I}_1$  is generated by the group of units  $\mathbb{I}_1^*$  of the algebra  $\mathbb{I}_1$  and the element  $\int$ . Moreover,  $\mathcal{L}(\mathbb{I}_1) = \bigsqcup_{i \in \mathbb{N}} \mathbb{I}_1^* \int^n$ , the disjoint union (Theorem 7.3.(1,2)). Similarly, the monoid  $\mathcal{R}(\mathbb{I}_1) := \{b \in \mathbb{I}_1 \mid ba = 1 \text{ for some } a \in \mathbb{I}_1\}$  of right invertible elements of the algebra  $\mathbb{I}_1$  is generated by the group  $\mathbb{I}_1^*$  and the element  $\partial$ . Moreover,  $\mathcal{R}(\mathbb{I}_1) = \bigsqcup_{i \in \mathbb{N}} \partial^n \mathbb{I}_1^*$ , the disjoint union (Theorem 7.3.(3,4)). For each left invertible element  $a \in \mathcal{L}(\mathbb{I}_1)$  (respectively, right invertible element  $b \in \mathcal{R}(\mathbb{I}_1)$ ) the set of its left (respectively, right) inverses is found (Theorem 7.5).

In Section 8, we introduce two algebras  $\widetilde{\mathbb{I}}_1$  and  $\widetilde{\mathbb{J}}_1$  that turn out to be the *largest right quotient ring*  $\text{Frac}_r(\mathbb{I}_1)$  and the *largest left quotient ring*  $\text{Frac}_l(\mathbb{I}_1)$  of the algebra  $\mathbb{I}_1$  (Theorem 9.7). The algebra  $\widetilde{\mathbb{I}}_1$  is the subalgebra of  $\text{End}_K(K[x])$  generated by the algebra  $\mathbb{I}_1$  and the (large) set

$$\mathbb{I}_1^0 := \mathbb{I}_1 \cap \text{Aut}_K(K[x]),$$

the set of all the elements of  $\mathbb{I}_1$  that are *invertible* linear maps in  $K[x]$ ; the group of units  $\mathbb{I}_1^*$  of the algebra  $\mathbb{I}_1$  is a small part of the monoid  $\mathbb{I}_1^0$ . In particular, the set  $\mathbb{I}_1^0$  is the *largest* (w.r.t. inclusion) *regular right Ore set* in  $\mathbb{I}_1$  but it is not a left Ore set of  $\mathbb{I}_1$ , and  $\widetilde{\mathbb{I}}_1 = \mathbb{I}_1 \mathbb{I}_1^0{}^{-1}$  (Theorem 9.7.(4)). The algebras  $\widetilde{\mathbb{I}}_1$  and  $\widetilde{\mathbb{J}}_1$  contain the only proper ideal,  $\mathcal{C}(\widetilde{\mathbb{I}}_1)$  and  $\mathcal{C}(\widetilde{\mathbb{J}}_1)$ , respectively (precisely the ideal of compact operators in  $\widetilde{\mathbb{I}}_1$  and  $\widetilde{\mathbb{J}}_1$  respectively). The factor algebras  $\widetilde{\mathbb{I}}_1/\mathcal{C}(\widetilde{\mathbb{I}}_1)$  and  $\widetilde{\mathbb{J}}_1/\mathcal{C}(\widetilde{\mathbb{J}}_1)$  are canonically isomorphic to the skew field of fractions  $\text{Frac}(A_1)$  of the first Weyl algebra  $A_1$  and its opposite skew field  $\text{Frac}(A_1)^{op}$  respectively (Theorem 8.3.(4) and Corollary 8.5.(4)). The simple (left and right) modules for the algebras  $\widetilde{\mathbb{I}}_1$  and  $\widetilde{\mathbb{J}}_1$  are classified, the groups of units of the algebras  $\widetilde{\mathbb{I}}_1$  and  $\widetilde{\mathbb{J}}_1$  are found (Proposition 9.1).

In Section 9, it is proved that the rings  $\widetilde{\mathbb{I}}_1$  and  $\widetilde{\mathbb{J}}_1$  are not  $\mathbb{I}_1$ -isomorphic but  $\mathbb{I}_1$ -*anti-isomorphic*. The sets of right regular, left regular and regular elements of the algebra  $\mathbb{I}_1$  are described (Lemma 9.3.(1), Corollary 9.4.(1) and Corollary 9.5).

**The Conjecture/Problem of Dixmier** (1968) [still open]: *is an algebra endomorphism of the Weyl algebra  $A_1$  an automorphism?* It turns out that an analogue of the Conjecture/Problem of Dixmier is true for the algebra  $\mathbb{I}_1$ :

- (Theorem 1.1, [14]) *Each algebra endomorphism of the algebra  $\mathbb{I}_1$  is an automorphism.*

The present paper is instrumental in proving this result.

The algebras  $\mathbb{I}_n := \mathbb{I}_1^{\otimes n}$  ( $n \geq 1$ ) of polynomial integro-differential operators are studied in detail in [12], their groups of automorphisms are found in [13].

## 2 Classification of simple $\mathbb{I}_1$ -modules

In this section, a classification of simple  $\mathbb{I}_1$ -modules is given (Theorem 2.1) and it is compared to a similar classification of simple modules over the Weyl algebra  $A_1$  (Theorem 2.2).

The algebra  $\mathbb{I}_1$  is generated by the elements  $\partial$ ,  $H := \partial x$  and  $\int$  (since  $x = \int H$ ) that satisfy the defining relations (Proposition 2.2, [12]):

$$\partial \int = 1, \quad [H, \int] = \int, \quad [H, \partial] = -\partial, \quad H(1 - \int \partial) = (1 - \int \partial)H = 1 - \int \partial.$$

The elements of the algebra  $\mathbb{I}_1$ ,

$$e_{ij} := \int^i \partial^j - \int^{i+1} \partial^{j+1}, \quad i, j \in \mathbb{N}, \quad (1)$$

satisfy the relations  $e_{ij}e_{kl} = \delta_{jk}e_{il}$  where  $\delta_{jk}$  is the Kronecker delta function. Notice that  $e_{ij} = \int^i e_{00} \partial^j$ . The matrices of the linear maps  $e_{ij} \in \text{End}_K(K[x])$  with respect to the basis  $\{x^{[s]} :=$

$\frac{x^s}{s!}\}_{s \in \mathbb{N}}$  of the polynomial algebra  $K[x]$  are the elementary matrices, i.e.,

$$e_{ij} * x^{[s]} = \begin{cases} x^{[i]} & \text{if } j = s, \\ 0 & \text{if } j \neq s. \end{cases}$$

Let  $E_{ij} \in \text{End}_K(K[x])$  be the usual matrix units, i.e.,  $E_{ij} * x^s = \delta_{js} x^i$  for all  $i, j, s \in \mathbb{N}$ . Then

$$e_{ij} = \frac{j!}{i!} E_{ij}, \quad (2)$$

$Ke_{ij} = KE_{ij}$ , and  $F := \bigoplus_{i,j \geq 0} Ke_{ij} = \bigoplus_{i,j \geq 0} KE_{ij} \simeq M_\infty(K)$ , the algebra (without 1) of infinite dimensional matrices. Using induction on  $i$  and the fact that  $\int^j e_{kk} \partial^j = e_{k+j, k+j}$ , we can easily prove that

$$\int^i \partial^i = 1 - e_{00} - e_{11} - \cdots - e_{i-1, i-1} = 1 - E_{00} - E_{11} - \cdots - E_{i-1, i-1}, \quad i \geq 1. \quad (3)$$

**A  $\mathbb{Z}$ -grading on the algebra  $\mathbb{I}_1$  and the canonical form of an integro-differential operator, [12].** The algebra  $\mathbb{I}_1 = \bigoplus_{i \in \mathbb{Z}} \mathbb{I}_{1,i}$  is a  $\mathbb{Z}$ -graded algebra ( $\mathbb{I}_{1,i} \mathbb{I}_{1,j} \subseteq \mathbb{I}_{1,i+j}$  for all  $i, j \in \mathbb{Z}$ ) where

$$\mathbb{I}_{1,i} = \begin{cases} D_1 \int^i = \int^i D_1 & \text{if } i > 0, \\ D_1 & \text{if } i = 0, \\ \partial^{|i|} D_1 = D_1 \partial^{|i|} & \text{if } i < 0, \end{cases} \quad (4)$$

the algebra  $D_1 := K[H] \oplus \bigoplus_{i \in \mathbb{N}} Ke_{ii}$  is a commutative non-Noetherian subalgebra of  $\mathbb{I}_1$ ,  $He_{ii} = e_{ii}H = (i+1)e_{ii}$  for  $i \in \mathbb{N}$  (notice that  $\bigoplus_{i \in \mathbb{N}} Ke_{ii}$  is the direct sum of non-zero ideals of  $D_1$ );  $(\int^i D_1)_{D_1} \simeq D_1$ ,  $\int^i d \mapsto d$ ;  $D_1(D_1 \partial^i) \simeq D_1$ ,  $d \partial^i \mapsto d$ , for all  $i \geq 0$  since  $\partial^i \int^i = 1$ . Notice that the maps  $\cdot \int^i : D_1 \rightarrow D_1 \int^i$ ,  $d \mapsto d \int^i$ , and  $\partial^i \cdot : D_1 \rightarrow \partial^i D_1$ ,  $d \mapsto \partial^i d$ , have the same kernel  $\bigoplus_{j=0}^{i-1} Ke_{jj}$ .

Each element  $a$  of the algebra  $\mathbb{I}_1$  is the unique finite sum

$$a = \sum_{i > 0} a_{-i} \partial^i + a_0 + \sum_{i > 0} \int^i a_i + \sum_{i,j \in \mathbb{N}} \lambda_{ij} e_{ij} \quad (5)$$

where  $a_k \in K[H]$  and  $\lambda_{ij} \in K$ . This is the *canonical form* of the polynomial integro-differential operator [12].

*Definition.* Let  $a \in \mathbb{I}_1$  be as in (5) and let  $a_F := \sum \lambda_{ij} e_{ij}$ . Suppose that  $a_F \neq 0$  then

$$\deg_F(a) := \min\{n \in \mathbb{N} \mid a_F \in \bigoplus_{i,j=0}^n Ke_{ij}\} \quad (6)$$

is called the *F-degree* of the element  $a$ ;  $\deg_F(0) := -1$ .

$$\text{Let } v_i := \begin{cases} \int^i & \text{if } i > 0, \\ 1 & \text{if } i = 0, \\ \partial^{|i|} & \text{if } i < 0. \end{cases}$$

Then  $\mathbb{I}_{1,i} = D_1 v_i = v_i D_1$  and an element  $a \in \mathbb{I}_1$  is the unique finite sum

$$a = \sum_{i \in \mathbb{Z}} b_i v_i + \sum_{i,j \in \mathbb{N}} \lambda_{ij} e_{ij} \quad (7)$$

where  $b_i \in K[H]$  and  $\lambda_{ij} \in K$ . So, the set  $\{H^j \partial^i, H^j, \int^i H^j, e_{st} \mid i \geq 1; j, s, t \geq 0\}$  is a  $K$ -basis for the algebra  $\mathbb{I}_1$ . The multiplication in the algebra  $\mathbb{I}_1$  is given by the rule:

$$\int H = (H-1) \int, \quad H \partial = \partial(H-1), \quad \int e_{ij} = e_{i+1,j}, \quad e_{ij} \int = e_{i,j-1}, \quad \partial e_{ij} = e_{i-1,j}, \quad e_{ij} \partial = \partial e_{i,j+1}.$$

$$He_{ii} = e_{ii}H = (i+1)e_{ii}, \quad i \in \mathbb{N},$$

where  $e_{-1,j} := 0$  and  $e_{i,-1} := 0$ .

**The ideal  $F$  of compact operators of  $\mathbb{I}_1$ .** Let  $V$  be an infinite dimensional vector space over a field  $K$ . A linear map  $\varphi \in \text{End}_K(V)$  is called a *compact* linear map/operator if  $\dim_K(\text{im}(\varphi)) < \infty$ . The set  $\mathcal{C} = \mathcal{C}(V)$  of all compact operators is a (two-sided) ideal of the algebra  $\text{End}_K(V)$ . The algebra  $\mathbb{I}_1$  has the only proper ideal

$$F = \bigoplus_{i,j \in \mathbb{N}} Ke_{ij} \simeq M_\infty(K),$$

the ideal of compact operators in  $\mathbb{I}_1$ ,  $F = \mathbb{I}_1 \cap \mathcal{C}(K[x])$  (Corollary 3.3),  $F^2 = F$ . The factor algebra  $\mathbb{I}_1/F$  is canonically isomorphic to the skew Laurent polynomial algebra  $B_1 := K[H][\partial, \partial^{-1}; \tau]$ ,  $\tau(H) = H+1$ , via  $\partial \mapsto \partial$ ,  $\int \mapsto \partial^{-1}$ ,  $H \mapsto H$  (where  $\partial^{\pm 1}\alpha = \tau^{\pm 1}(\alpha)\partial^{\pm 1}$  for all elements  $\alpha \in K[H]$ ). The algebra  $B_1$  is canonically isomorphic to the (left and right) localization  $A_{1,\partial}$  of the Weyl algebra  $A_1$  at the powers of the element  $\partial$  (notice that  $x = \partial^{-1}H$ ). Therefore, they have the common skew field of fractions,  $\text{Frac}(A_1) = \text{Frac}(B_1)$ , the *first Weyl skew field*. The algebra  $B_1$  is a subalgebra of the skew Laurent polynomial algebra  $\mathcal{B}_1 := K(H)[\partial, \partial^{-1}; \tau]$  where  $K(H)$  is the field of rational functions over the field  $K$  in  $H$ . The algebra  $\mathcal{B}_1 = S^{-1}B_1$  is the left and right localization of the algebra  $B_1$  at the multiplicative set  $S = K[H] \setminus \{0\}$ . The algebra  $\mathcal{B}_1$  is a *noncommutative Euclidean domain*, i.e., the left and right division algorithms with remainder hold with respect to the length function  $l$  on  $B_1$ :

$$l(\alpha_m \partial^m + \alpha_{m+1} \partial^{m+1} + \cdots + \alpha_n \partial^n) = n - m$$

where  $\alpha_i \in K(H)$ ,  $\alpha_m \neq 0$ ,  $\alpha_n \neq 0$ , and  $m < \cdots < n$ . In particular, the algebra  $\mathcal{B}_1$  is a principal left and right ideal domain. A  $\mathcal{B}_1$ -module  $M$  is simple iff  $M \simeq \mathcal{B}_1/\mathcal{B}_1 b$  for some irreducible element  $b \in \mathcal{B}_1$ , and  $\mathcal{B}_1/\mathcal{B}_1 b \simeq \mathcal{B}_1/\mathcal{B}_1 c$  iff the elements  $b$  and  $c$  are *similar* (that is, there exists an element  $d \in \mathcal{B}_1$  such that 1 is the greatest common right divisor of  $c$  and  $d$ , and  $bd$  is the least common left multiple of  $c$  and  $d$ ).

**The involution  $*$  on the algebra  $\mathbb{I}_1$ .** The algebra  $\mathbb{I}_1$  admits the involution  $*$  over the field  $K$ :

$$\partial^* = \int, \quad \int^* = \partial \quad \text{and} \quad H^* = H,$$

i.e., it is a  $K$ -algebra *anti-isomorphism*  $((ab)^* = b^*a^*)$  such that  $a^{**} = a$ . Therefore, the algebra  $\mathbb{I}_1$  is *self-dual*, i.e., it is isomorphic to its opposite algebra  $\mathbb{I}_1^{op}$ . As a result, the left and right properties of the algebra  $\mathbb{I}_1$  are the same. Clearly,  $e_{ij}^* = e_{ji}$  for all  $i, j \in \mathbb{N}$ , and so  $F^* = F$ .

**A classification of simple  $\mathbb{I}_1$ -modules.** Since the field  $K$  has characteristic zero, the group  $\langle \tau \rangle \simeq \mathbb{Z}$  acts freely on the set  $\text{Max}(K[H])$  of maximal ideals of the polynomial algebra  $K[H]$ . That is, for each maximal ideal  $\mathfrak{p} \in \text{Max}(K[H])$ , its orbit  $\mathcal{O}(\mathfrak{p}) := \{\tau^i(\mathfrak{p}) \mid i \in \mathbb{Z}\}$  contains infinitely many elements. For two elements  $\tau^i(\mathfrak{p})$  and  $\tau^j(\mathfrak{p})$  of the orbit  $\mathcal{O}(\mathfrak{p})$ , we write  $\tau^i(\mathfrak{p}) < \tau^j(\mathfrak{p})$  if  $i < j$ . Let  $\text{Max}(K[H])/\langle \tau \rangle$  be the set of all  $\langle \tau \rangle$ -orbits in  $\text{Max}(K[H])$ . If  $K$  is an algebraically closed field then  $\mathfrak{p} = (H - \lambda)$ , for some scalar  $\lambda \in K$ ,  $\text{Max}(K[H]) \simeq K$ , and  $\text{Max}(K[H])/\langle \tau \rangle \simeq K/\mathbb{Z}$ .

For elements  $\alpha, \beta \in K[H]$ , we write  $\alpha < \beta$  if for all maximal ideals  $\mathfrak{p}, \mathfrak{q} \in \text{Max}(K[H])$  that belong to the same orbit and such that  $\alpha \in \mathfrak{p}$  and  $\beta \in \mathfrak{q}$  we have  $\mathfrak{p} < \mathfrak{q}$ . If  $K$  is an algebraically closed field then  $p < q$  iff  $\lambda - \mu > 0$  for all roots  $\lambda$  and  $\mu$  of the polynomials  $p$  and  $q$  respectively (if they exist) such that  $\lambda - \mu \in \mathbb{Z}$ .

*Definition.* An element  $b = \partial^{-m}\beta_{-m} + \cdots + \beta_0 \in B_1$ , where all  $\beta_i \in K[H]$ ,  $\beta_{-m} \neq 0$ ,  $\beta_0 \neq 0$ , and  $m > 0$ , is called a *normal* element if  $\beta_0 < \beta_{-m}$ .

Let the element  $b = \partial^{-m}\beta_{-m} + \cdots + \beta_0 \in B_1$  be such as in the Definition above but not necessarily normal. Then *there exist polynomials  $\alpha, \beta \in K[H]$  such that the element  $\beta b \alpha^{-1} \in B_1$  is normal*. For example,

$$\alpha = \prod \{\tau^i(\beta_0) \mid -s \leq i \leq 0\}, \quad \beta = \prod \{\tau^j(\beta_0) \mid -m-s \leq j \leq 1\}$$

where  $s \in \mathbb{N}$  such that  $\tau^s(\beta_0) < \beta_{-m}$  and  $\tau^{-s}(\beta_0) < \beta_0$  (Proposition 16, [6]).

For an algebra  $A$ , let  $\widehat{A}$  be the set of isomorphism classes of simple  $A$ -modules and, for a simple  $A$ -module  $M$ , let  $[M] \in \widehat{A}$  be its isomorphism class. For an isomorphism invariant property  $\mathcal{P}$  of simple  $A$ -modules, let

$$\widehat{A}(\mathcal{P}) := \{[M] \in \widehat{A} \mid M \text{ has property } \mathcal{P}\}.$$

The *socle*  $\text{soc}(M)$  of a module  $M$  is the sum of all the simple submodules if they exist and zero otherwise. Since the algebra  $\mathbb{I}_1$  contains the Weyl algebra  $A_1$ , which is a simple infinite dimensional algebra, each nonzero  $\mathbb{I}_1$ -module is necessarily an *infinite dimensional* module.

The algebra  $B_1 = K[H](\tau, 1)$ ,  $\tau(H) = H + 1$ , is an example of a generalized Weyl algebra (GWA)  $A = D(\sigma, a)$  over a *Dedekind* domain  $D$  which is in the case of the algebra  $B_1$  is the polynomial algebra  $K[H]$ . Many popular algebras of small Gelfand-Kirillov dimension are examples of  $A$  (eg, the Weyl algebra  $A_1$ , the *quantum plane*  $\Lambda = K\langle x, y \mid xy = \lambda yx \rangle$ ,  $\lambda \in K^*$ ; the *quantum Weyl algebra*  $A_1(q) = K\langle x, \partial \mid \partial x - qx\partial = 1 \rangle$ ,  $q \in K \setminus \{0, 1\}$ , all prime infinite dimensional factor algebras of  $Usl(2)$ , etc). A classification of simple  $A$ -modules is obtained in [4, 5, 6, 7].

Since the algebra  $B_1$  is a factor algebra of  $\mathbb{I}_1$ , there is the tautological embedding

$$\widehat{B}_1 \rightarrow \widehat{\mathbb{I}}_1, \quad [M] \mapsto [M].$$

Therefore,  $\widehat{B}_1 \subseteq \widehat{\mathbb{I}}_1$ .

**Theorem 2.1** (Classification of simple  $\mathbb{I}_1$ -modules)

1.  $\widehat{\mathbb{I}}_1 = \{[K[x]]\} \amalg \widehat{B}_1$ .
2. The set  $\widehat{B}_1 = \widehat{B}_1(K[H] - \text{torsion}) \amalg \widehat{B}_1(K[H] - \text{torsion free})$  is the disjoint union where  $\widehat{B}_1(K[H] - \text{torsion}) := \{[M] \in \widehat{B}_1 \mid S^{-1}M = 0\}$ ,  $\widehat{B}_1(K[H] - \text{torsion free}) := \{[M] \in \widehat{B}_1 \mid S^{-1}M \neq 0\}$ , and  $S := K[H] \setminus \{0\}$ .
3. The map
$$\text{Max}(K[H])/\langle \tau \rangle \rightarrow \widehat{B}_1(K[H] - \text{torsion}), \quad [\mathfrak{p}] \mapsto [B_1/B_1\mathfrak{p}],$$
is a bijection with the inverse map  $[N] \mapsto \text{supp}(N) := \{\mathfrak{p} \in \text{Max}(K[H]) \mid \mathfrak{p} \cdot n_{\mathfrak{p}} = 0 \text{ for some } 0 \neq n_{\mathfrak{p}} \in N\}$ .
4. (Classification of simple  $K[H]$ -torsion free  $\mathbb{I}_1$ -modules)

(a) The map

$$\widehat{B}_1(K[H] - \text{torsion free}) \rightarrow \widehat{B}_1, \quad [M] \mapsto [S^{-1}M],$$

is a bijection with the inverse map  $\text{soc}_{B_1} : [N] \mapsto [\text{soc}_{B_1}(N)]$  where  $\text{soc}_{B_1}(N)$  is the socle of the  $B_1$ -module  $N$ .

- (b) Let  $b = \partial^{-m}\beta_{-m} + \cdots + \beta_0 \in B_1$  be a normal element of the algebra  $B_1$  which is an irreducible element of the algebra  $B_1$  where all  $\beta_i \in K[H]$ ,  $\beta_{-m} \neq 0$ ,  $\beta_0 \neq 0$ , and  $m > 0$ . Then  $B_1/B_1 \cap B_1b$  is a  $K[H]$ -torsion free simple  $B_1$ -module. Up to isomorphism, every  $K[H]$ -torsion free simple  $B_1$ -module arises in this way, and from  $b$  which is unique up to similarity.
- (c) An  $\mathbb{I}_1$ -module  $M$  is simple  $K[H]$ -torsion free iff  $M \simeq (B_1\alpha + B_1b)/B_1b \simeq B_1/B_1 \cap B_1b\alpha^{-1}$  for some irreducible in  $B_1$  element  $b \in B_1$  such that the element  $\beta b \alpha^{-1} \in B_1$  is normal where  $\alpha, \beta \in K[H]$ .

*Proof.* 1. Statement 1 follows from the facts that  $K[x]$  is the only (up to isomorphism) faithful simple  $\mathbb{I}_1$ -module (Proposition 6.1, [12]),  $F$  is the only proper ideal of the algebra  $\mathbb{I}_1$  and  $B_1 = \mathbb{I}_1/F$ .

2. Statement 2 is obvious.

3. Statement 3 is a particular case of Theorem 11, [6].

4. Statement 4 is a particular case of Theorem 17, [6].  $\square$

Let  $\mathfrak{p} \in \text{Max}(K[H])$ . Then the simple  $\mathbb{I}_1$ -module  $B_1/B_1\mathfrak{p} \simeq \bigoplus_{i \in \mathbb{Z}} \partial^i K[H]/\mathfrak{p}$  is a semi-simple  $K[H]$ -module since  ${}_{K[H]}(\partial^i K[H]/\mathfrak{p}) \simeq K[H]/\tau^i(\mathfrak{p})$ .

A classification of simple  $A_1$ -modules was given by Block, [17] and [18].

**Relationships between the sets of simple  $\mathbb{I}_1$ -modules and  $A_1$ -modules.** We will see that the algebras  $\mathbb{I}_1$  has “one less” simple module than the Weyl algebra  $A_1$  (Theorem 2.2.(3)). The algebra  $B_1$  is a left (but not right) localization of the algebra  $\mathbb{I}_1$  at the powers of the element  $\partial$  [12],  $B_1 = S_\partial^{-1}\mathbb{I}_1$  where  $S_\partial := \{\partial^i \mid i \in \mathbb{N}\}$ . The algebra  $B_1 = S_\partial^{-1}A_1$  is the left and right localization of the Weyl algebra  $A_1$  at  $S_\partial$ . It follows at once from the fact that the  $\mathbb{I}_1$ -module  $\mathbb{I}_1/\mathbb{I}_1\partial = K[x]$  is simple and  $\partial$ -torsion (i.e.,  $S_\partial^{-1}K[x] = 0$ ) that

$$\widehat{\mathbb{I}}_1(\partial - \text{torsion}) = \{[K[x]]\}.$$

Similarly, the  $A_1$ -module  $A_1/A_1\partial = K[x]$  is simple and  $\partial$ -torsion. Then

$$\widehat{A}_1(\partial - \text{torsion}) = \{[K[x]]\}.$$

The algebra  $\mathcal{B}_1 = K(H)[\partial, \partial^{-1}; \tau]$  is a left and right localization of the algebras  $\mathbb{I}_1$  and  $A_1$  at  $S := K[H] \setminus \{0\}$ ,

$$\mathcal{B}_1 = S^{-1}\mathbb{I}_1 = S^{-1}A_1.$$

Therefore, the sets of simple  $\mathbb{I}_1$ - and  $A_1$ -modules can be represented as the disjoint unions of  $K[H]$ -torsion and  $K[H]$ -torsion free modules:

$$\begin{aligned} \widehat{\mathbb{I}}_1 &= \widehat{\mathbb{I}}_1(K[H] - \text{torsion}) \coprod \widehat{\mathbb{I}}_1(K[H] - \text{torsion free}), \\ \widehat{A}_1 &= \widehat{A}_1(K[H] - \text{torsion}) \coprod \widehat{A}_1(K[H] - \text{torsion free}), \end{aligned}$$

where

$$\begin{aligned} \widehat{\mathbb{I}}_1(K[H] - \text{torsion}) &= \{[K[x]]\} \coprod \widehat{B}_1(K[H] - \text{torsion}), \\ \widehat{\mathbb{I}}_1(K[H] - \text{torsion free}) &= \widehat{B}_1(K[H] - \text{torsion free}), \\ \widehat{A}_1(K[H] - \text{torsion}) &= \{[A_1/A_1x]\} \coprod \{[K[x]]\} \coprod \widehat{A}'_1(K[H] - \text{torsion}), \end{aligned}$$

where the set  $\widehat{A}'_1(K[H] - \text{torsion})$  is obtained from the set  $\widehat{A}_1(K[H] - \text{torsion})$  by deleting its two elements  $[A_1/A_1x]$  and  $[K[x]]$ .

Recall that the Weyl algebra  $A_1$  is a GWA  $K[H](\sigma, H)$ ,  $\sigma(H) = H - 1$ , over the Dedekind domain  $K[H]$ . There is a similar to Theorem 2.1 classification of simple  $A_1$ -modules [4], [5], [6]. Comparing it to Theorem 2.1, the next theorem follows.

**Theorem 2.2** 1. The map  $\widehat{A}'_1(K[H] - \text{torsion}) \rightarrow \widehat{B}_1(K[H] - \text{torsion})$ ,  $[M] \mapsto [S_\partial^{-1}M]$ , is a bijection with the inverse map  $[N] \mapsto [A_1N]$ .

2. The map  $\widehat{A}_1(K[H] - \text{torsion free}) \rightarrow \widehat{B}_1(K[H] - \text{torsion free})$ ,  $[M] \mapsto [S_\partial^{-1}M]$ , is a bijection with the inverse map  $[N] \mapsto [\text{soc}_{A_1}N]$ .

3. Combining statements 1 and 2, the map

$$\widehat{A}_1 \setminus \{[A_1/A_1x]\} \rightarrow \widehat{\mathbb{I}}_1 = \{[K[x]]\} \coprod \widehat{B}_1, \quad [K[x]] \mapsto [K[x]], \quad [M] \mapsto [S_\partial^{-1}M],$$

is a bijection with the inverse map  $[K[x]] \mapsto [K[x]]$  and  $[N] \mapsto [\text{soc}_{A_1}N]$  where  $[N] \in \widehat{B}_1$ .



### 3 The Strong Compact-Fredholm Alternative

In this section, the Strong Compact-Fredholm Alternative is proved for the algebra  $\mathbb{I}_1$  (Theorem 3.1), it is shown that the endomorphism algebra of each simple  $\mathbb{I}_1$ -module is a finite dimensional division algebra (Theorem 3.4). Various indices are introduced for non-compact integro-differential operators, the  $M$ -index and the length index, and we show that they are invariant under addition of compact operator (Lemma 3.5 and Lemma 3.7).

**The (Strong) Compact-Fredholm Alternative.** Let  $\mathcal{F} = \mathcal{F}[K]$  be the family of all  $K$ -linear maps with finite dimensional kernel and cokernel, such maps are called the *Fredholm linear maps/operators*. So,  $\mathcal{F}$  is the family of *Fredholm linear maps/operators*. For vector spaces  $V$  and  $U$ , let  $\mathcal{F}(V, U)$  be the set of all the linear maps from  $V$  to  $U$  with finite dimensional kernel and cokernel and  $\mathcal{F}(V) := \mathcal{F}(V, V)$ . So,  $\mathcal{F} = \coprod_{V, U} \mathcal{F}(V, U)$  is the disjoint union.

*Definition.* For a linear map  $\varphi \in \mathcal{F}$ , the integer

$$\text{ind}(\varphi) := \dim \ker(\varphi) - \dim \text{coker}(\varphi)$$

is called the *index* of the map  $\varphi$ .

*Example.* Note that  $\partial, \int \in \mathbb{I}_1 \subset \text{End}_K(K[x])$ . Then

$$\text{ind}(\partial^i) = i \text{ and } \text{ind}\left(\int^i\right) = -i, \quad i \geq 1. \quad (8)$$

It is well-known that for any two linear maps  $M \xrightarrow{a} N \xrightarrow{b} L$  there is the long exact sequence of vector spaces

$$0 \rightarrow \ker(a) \rightarrow \ker(ba) \rightarrow \ker(b) \rightarrow \text{coker}(a) \rightarrow \text{coker}(ba) \rightarrow \text{coker}(b) \rightarrow 0. \quad (9)$$

Let  $\mathcal{C}$  be the family of  $K$ -linear maps with finite dimensional image, such maps are called the *compact linear maps/operators*. For vector spaces  $V$  and  $U$ , let  $\mathcal{C}(V, U)$  be the set of all the compact linear maps from  $V$  to  $U$ . So,  $\mathcal{C} = \coprod_{V, U} \mathcal{C}(V, U)$  is the disjoint union. If  $V = U$  we write  $\mathcal{C}(V) := \mathcal{C}(V, V)$ .

*Definitions.* Let  $A$  be an algebra and  $M$  be its module. We say that, for the  $A$ -module  $M$ , the *Compact-Fredholm Alternative* holds if, for each element  $a \in A$ , the linear map  $a_M : M \rightarrow M$ ,  $m \mapsto am$ , is either compact or Fredholm. We say that for the algebra  $A$  the *Compact-Fredholm Alternative* holds if for each *simple*  $A$ -module it does; and the *Strong Compact-Fredholm Alternative* holds if, in addition, for each element  $a \in A$  either the linear maps  $a_M$  are compact for all simple  $A$ -modules  $M$  or, otherwise, they are all Fredholm. In the first and the second case the element  $a$  is called a *compact* and *Fredholm* element respectively, and the sets of all compact and Fredholm elements of the algebra  $A$  are denoted by  $\mathcal{C}_A$  and  $\mathcal{F}_A$  respectively; then

$$A = \mathcal{C}_A \coprod \mathcal{F}_A$$

is the disjoint union (for the algebra  $A$  with Strong Compact-Fredholm Alternative). Clearly,  $\mathcal{C}_A$  is an ideal of the algebra  $A$  and  $\mathcal{C}_A \supseteq \text{rad}(A)$ , the Jacobson radical of  $A$ . By (9),  $\mathcal{F}_A$  is a multiplicative monoid that contains the group of units of the algebra  $A$ .

The next theorem shows that each non-compact integro-differential operator has only finitely many linearly independent solutions in each (left or right)  $\mathbb{I}_1$ -module of finite length. An analogous result is known for the Weyl algebra  $A_1$  [25] (for simple  $A_1$ -modules). Notice that each nonzero element of the Weyl algebra  $A_1$  is a non-compact operators, i.e.,  $A_1 \cap \mathcal{C} = \{0\}$ .

**Theorem 3.1** (Strong Compact-Fredholm Alternative) *For the algebra  $\mathbb{I}_1$  the Strong Compact-Fredholm Alternative holds (for left and right simple  $\mathbb{I}_1$ -modules);  $\mathcal{C}_{\mathbb{I}_1} = F$  and  $\mathcal{F}_{\mathbb{I}_1} = \mathbb{I}_1 \setminus F$ . Moreover,*

1. *Let  $a \in \mathbb{I}_1$ ,  ${}_1M$  be a nonzero  $\mathbb{I}_1$ -module of finite length and  $a_M : M \rightarrow M$ ,  $m \mapsto am$ . Then*
  - (a)  $\dim_K(\ker(a_M)) < \infty$  iff  $\dim_K(\operatorname{coker}(a_M)) < \infty$  iff  $a \notin F$ .
  - (b) *If, in addition,  ${}_1M$  is simple then  $\dim_K(\operatorname{im}(a_M)) < \infty$  iff  $a \in F$ .*
2. *Let  $a \in \mathbb{I}_1$ ,  $N_{\mathbb{I}_1}$  be a nonzero right  $\mathbb{I}_1$ -module of finite length and  $a_N : N \rightarrow N$ ,  $n \mapsto na$ . Then*
  - (a)  $\dim_K(\ker(a_N)) < \infty$  iff  $\dim_K(\operatorname{coker}(a_N)) < \infty$  iff  $a \notin F$ .
  - (b) *If, in addition,  $N_{\mathbb{I}_1}$  is simple then  $\dim_K(\operatorname{im}(a_N)) < \infty$  iff  $a \in F$ .*

*Proof.* 1(a). Using induction on the length of the module  $M$  and the Snake Lemma we see that statement 1(a) holds iff it does for each simple  $\mathbb{I}_1$ -module. If  $a \in F$  then, by Theorem 2.1, for each simple  $\mathbb{I}_1$ -module  $M$ , the kernel and the cokernel of the map  $a_M$  are infinite dimensional spaces. It remains to show that if  $a \notin F$  then the kernel and the cokernel of the map  $a_M$  are finite dimensional spaces. First we consider the case when  $M = K[x]$ . The algebra  $\mathbb{I}_1$  is a  $\mathbb{Z}$ -graded algebra. The usual  $\mathbb{N}$ -grading of the polynomial algebra  $K[x] = \bigoplus_{i \geq 0} Kx^i$  is also a  $\mathbb{Z}$ -grading of the  $\mathbb{I}_1$ -module  $K[x]$ . It determines the filtration  $K[x] = \bigcup_{i \geq 0} K[x]_{\leq i}$  where  $K[x]_{\leq i} := \bigoplus_{j=0}^i Kx^j$ . Clearly, when  $a = \partial^i$  or  $a = \int^i$ , the map  $a_{K[x]}$  has finite dimensional kernel and cokernel. Using (9) and (5), we may assume that all the polynomials  $a_i$ ,  $i > 0$ , in the decomposition (5) are equal to zero and  $a_0 \neq 0$ . Fix a natural number  $m$  such that all  $\lambda_{ij} = 0$  for all  $i, j \geq m$  and  $a_0(H) * x^l = a(l+1)x^l \neq 0$  for all  $l \geq m$ . Then

$$aK[x]_{\leq l} \subseteq K[x]_{\leq l} \text{ for all } l \geq m,$$

and the element  $a$  acts as a bijection on the factor space  $K[x]/K[x]_{\leq m}$ . This implies that the map  $a_{K[x]}$  has finite dimensional kernel and cokernel.

Suppose that  $M \not\simeq K[x]$ , i.e.,  $M$  is a simple  $B_1$ -module (Theorem 2.1.(1)). The algebra  $B_1$  is an example of the generalized Weyl algebra (GWA)  $A = K[H](\tau, u)$  over the Dedekind domain  $K[H]$  where  $\tau(H) = H + 1$  and  $u = 1$ . Statement 2 was established for the GWA  $A$  over an algebraically closed field and for simple  $A$ -modules (Theorem 4, [4]). Using the classification of simple  $\mathbb{I}_1$ -modules (Theorem 2.1) and the classification of simple  $A$ -modules in [4] (or [6]) we see that, for each simple  $B_1$ -module  $M$ , the  $B_1(\overline{K})$ -module  $B_1(\overline{K}) \otimes_{B_1} M = \overline{K} \otimes_K M$  has finite length (where  $\overline{K}$  is the algebraic closure of the field  $K$ ), and the result follows from Theorem 4, [4].

There is a more elementary way to show that the  $B_1(\overline{K})$ -module  $\overline{K} \otimes_K M$  has finite length. For, notice that over an arbitrary field (i) the algebra  $B_1$  is a simple Noetherian domain of Gelfand-Kirillov dimension 2 which is an almost commutative algebra with respect to the standard filtration  $\mathcal{V}$  associated with the generators  $\partial$ ,  $\partial^{-1}$  and  $H$  (i.e., the associated graded algebra is a commutative finitely generated algebra); (ii) the Gelfand-Kirillov dimension of each simple  $B_1$ -module is 1; (iii) every finitely generated  $B_1$ -module of Gelfand-Kirillov dimension 1 has finite length which does not exceed the multiplicity of the module (with respect to the filtration  $\mathcal{V}$ ); (iv) since the Gelfand-Kirillov dimension and the multiplicity are invariant under field extensions, by (ii) and (iii), the  $B_1(\overline{K})$ -module  $\overline{K} \otimes_K M$  has finite length.

1(b). ( $\Rightarrow$ ) Each simple  $\mathbb{I}_1$ -module  $M$  is infinite dimensional. So, if  $\dim_K(\operatorname{im}(a_M)) < \infty$  then necessarily  $a \in F$ , by statement 1(a).

( $\Leftarrow$ ) If  $a \in F$  then, clearly,  $\dim_K(\operatorname{im}(a_M)) < \infty$  for all simple  $\mathbb{I}_1$ -modules  $M$ : this is obvious when  $M \simeq {}_1K[x]$  but if  $M \not\simeq {}_1K[x]$  then  $a_M = 0$  (Theorem 2.1.(1)).

2. In view of the involution  $*$  on the algebra  $\mathbb{I}_1$ , statement 2 follows from statement 1.  $\square$

**Corollary 3.2**  $(\mathcal{C}_{\mathbb{I}_n})^* = \mathcal{C}_{\mathbb{I}_n}$  and  $(\mathcal{F}_{\mathbb{I}_1})^* = \mathcal{F}_{\mathbb{I}_1}$ , i.e., the involution  $*$  preserves the compact and Fredholm elements of the algebra  $\mathbb{I}_1$ .

*Proof.* This follows from  $F^* = F$ ,  $\mathcal{C}_{\mathbb{I}_1} = F$  and  $\mathcal{F}_{\mathbb{I}_1} = \mathbb{I}_1 \setminus F$ .  $\square$

**Corollary 3.3**  $\mathbb{I}_n \cap \mathcal{C}(K[x]) = \mathcal{C}_{\mathbb{I}_1} = F$ .

*Proof.* By the very definition,  $\mathbb{I}_1 \subseteq \text{End}_K(K[x])$ . Since  $F \subseteq \mathcal{C}(K[x])$  and  $(\mathbb{I}_1 \setminus F) \cap \mathcal{C}(K[x]) = \emptyset$  (by Theorem 3.1), the first equality follows. The second equality is obvious (Theorem 3.1).  $\square$

**The endomorphism algebras of simple  $\mathbb{I}_1$ -modules are finite dimensional.** The endomorphism algebra  $\text{End}_A(M)$  of a simple module over an algebra  $A$  is a division algebra.

**Theorem 3.4** 1.  $\dim_K(\text{End}_{\mathbb{I}_1}(M)) < \infty$  for all simple  $\mathbb{I}_1$ -modules  $M$ .

2.  $\text{End}_{\mathbb{I}_1}(K[x]) \simeq K$ .

3.  $\text{End}_{\mathbb{I}_1}(B_1/B_1\mathfrak{p}) \simeq K[H]/\mathfrak{p}$  for all  $\mathfrak{p} \in \text{Max}(K[H])$ .

*Proof.* 2.  $\text{End}_{\mathbb{I}_1}(K[x]) \simeq \text{End}_{\mathbb{I}_1}(\mathbb{I}_1/\mathbb{I}_1\partial) \simeq \ker_{K[x]}(\partial) = K$ .

3. Recall that the  $K[H]$ -module  $B_1/B_1\mathfrak{p} = \bigoplus_{i \in \mathbb{Z}} \partial^i K[H]/\mathfrak{p}$  is the direct sum of simple pairwise non-isomorphic  $K[H]$ -modules  $_{K[H]}(\partial^i K[H]/\mathfrak{p}) \simeq K[H]/\tau^i(\mathfrak{p})$ . Therefore,  $\text{End}_{\mathbb{I}_1}(B_1/B_1\mathfrak{p}) \simeq \text{ann}_{B_1/B_1\mathfrak{p}}(\mathfrak{p}) = K[H]/\mathfrak{p}$ .

1. By Theorem 2.1 and statements 2 and 3, it remains to show that the endomorphism algebra of the  $\mathbb{I}_1$ -module  $M$  is finite dimensional in the case when  $M$  is  $K[H]$ -torsion free, i.e.,  $M \simeq B_1/B_1 \cap \mathcal{B}_1 b$  where the element  $b \in B_1$  is as in Theorem 2.1.(4). Since  $\text{End}_{\mathbb{I}_1}(M) \simeq \text{ann}_M(B_1 \cap \mathcal{B}_1 b) \subseteq \ker_M(b)$  and  $b \notin F$ , the algebra  $\text{End}_{\mathbb{I}_1}(M)$  is finite dimensional by Theorem 3.1.(1a).  $\square$

**The  $M$ -index and its properties.**

*Definition.* For each element  $a \in \mathbb{I}_1 \setminus F$  and a (left or right)  $\mathbb{I}_1$ -module  $M$  of finite length, we define the  $M$ -index of the element  $a$ ,

$$\text{ind}_M(a) := \dim_K(\ker(a_M)) - \dim_K(\text{coker}(a_M)).$$

The set  $\mathbb{I}_1 \setminus F$  is a multiplicative monoid. For all elements  $a, b \in \mathbb{I}_1 \setminus F$ ,

$$\text{ind}_M(ab) = \text{ind}_M(a) + \text{ind}_M(b). \quad (10)$$

This follows from (9). Therefore, the  $M$ -index  $\text{ind}_M : \mathbb{I}_1 \setminus F \rightarrow \mathbb{Z}$  is a monoid homomorphism. The next lemma shows that the  $M$ -index is invariant under addition of compact operator.

**Lemma 3.5** Let  $a \in \mathbb{I}_1 \setminus F$  and  $f \in F$ . Then  $\text{ind}_M(a+f) = \text{ind}_M(a)$  for all left or right  $\mathbb{I}_1$ -modules  $M$  of finite length. i.e.,  $\mathcal{F}_{\mathbb{I}_1} + \mathcal{C}_{\mathbb{I}_1} \subseteq \mathcal{F}_{\mathbb{I}_1}$ .

*Proof.* Note that  $\partial^i f = 0$  for all  $i \gg 0$ , and  $\partial^i, \partial^i a \notin F$ . Then, by (10),

$$\text{ind}_M(\partial^i) + \text{ind}_M(a) = \text{ind}_M(\partial^i a) = \text{ind}_M(\partial^i(a+f)) = \text{ind}_M(\partial^i) + \text{ind}_M(a+f),$$

hence  $\text{ind}_M(a) = \text{ind}_M(a+f)$ .  $\square$

Let  $\sigma$  be an automorphism of an algebra  $A$  and  $M$  be an  $A$ -module. The *twisted*  $A$ -module  ${}^\sigma M$  by the automorphism  $\sigma$  coincides with  $M$  as a vector space but the  $A$ -module structure on  ${}^\sigma M$  is given by the rule  $a \cdot m = \sigma(a)m$  for all elements  $a \in A$  and  $m \in M$ . For all elements  $a \in \mathbb{I}_1 \setminus F$ , an automorphism  $\sigma \in \text{Aut}_{K\text{-alg}}(\mathbb{I}_1)$  and an  $\mathbb{I}_1$ -module  $M$  of finite length (Theorem 3.1),

$$\text{ind}_M(\sigma(a)) = \text{ind}_{{}^\sigma M}(a). \quad (11)$$

If, in addition,  ${}^\sigma_{\mathbb{I}_1} M \simeq M$  for some automorphism  $\sigma$ , then

$$\text{ind}_M(\sigma(a)) = \text{ind}_M(a) \text{ for all } a \in \mathbb{I}_1 \setminus F. \quad (12)$$

If  ${}^\tau_{\mathbb{I}_1} M \not\simeq M$  then, in general, the equality (12) does not hold and this observation is a very effective tool in proving that two modules are non-isomorphic.

*Example.* For each  $\lambda \in K^*$ , the  $\mathbb{I}_1$ -module  $V_\lambda := B_1/B_1(\partial - \lambda)$  is simple: the  $K[H]$ -module homomorphism  $K[H] \rightarrow V_\lambda$ ,  $\alpha \mapsto \alpha + B_1(\partial - \lambda)$ , is obviously an epimorphism; therefore it is a  $K[H]$ -module isomorphism as each nonzero  $B_1$ -module is infinite dimensional and each proper factor module of the  $K[H]$ -module  $K[H]$  is finite dimensional; then  $V_\lambda$  is a simple  $B_1$ -module such that  ${}_{K[H]}V_\lambda \simeq K[H]$ . For each  $\lambda \in K^*$ , there is an automorphism  $t_\lambda$  of the algebra  $\mathbb{I}_1$  given by the rule

$$t_\lambda(\int) = \lambda \int, \quad t_\lambda(\partial) = \lambda^{-1}\partial, \quad t_\lambda(H) = H.$$

Since  $t_\lambda(e_{ij}) = \lambda^{i-j}e_{ij}$ ,  $t_\lambda(F) = F$ ,  $t_\lambda$  induces the automorphism  $t_\lambda$  of the factor algebra  $B_1 = \mathbb{I}_1/F$  by the rule  $t_\lambda(\partial) = \lambda^{-1}\partial$  and  $t_\lambda(H) = H$ . Since  $t_\lambda t_\mu = t_{\lambda\mu}$  for all  $\lambda, \mu \in K^*$ , the *algebraic torus*  $\mathbb{T}^1 := \{t_\lambda \mid \lambda \in K^*\} \simeq K^*$  is a subgroup of the groups  $\text{Aut}_{K\text{-alg}}(\mathbb{I}_1)$  and  $\text{Aut}_{K\text{-alg}}(B_1)$ . Clearly,  $V_\lambda \simeq {}^{t_\lambda^{-1}}V_1$  for all  $\lambda \in K^*$  (since  $t_{\lambda^{-1}}(\partial - \lambda) = \lambda(\partial - 1)$ ). The simple  $\mathbb{I}_1$ -modules  $\{V_\lambda \mid \lambda \in K^*\}$  is the  $\mathbb{T}^1$ -orbit of the module  $V_1$ . We aim to show that

$${}_{\mathbb{I}_1}V_\lambda \simeq {}_{\mathbb{I}_1}V_\mu \text{ iff } \lambda = \mu. \quad (13)$$

Without loss of generality we may assume that  $\mu = 1$ . Then the result follows at once from the fact that

$$\text{ind}_{V_1}(\partial - \lambda) = \begin{cases} 1 & \text{if } \lambda = 1, \\ 0 & \text{otherwise.} \end{cases} \quad (14)$$

*Proof of (14).* When we identify the  $\mathbb{I}_1$ -module  $V_1$  with  $K[H]$  (as we did above), the action of the element  $\partial - \lambda$  on  $V_1$  is identified with the action of the linear map  $\tau - \lambda$  on  $K[H]$  where  $\tau(H) = H + 1$  since  $\partial \cdot 1 = 1$ . The map  $\tau - 1$  is surjective with kernel  $K$ , then  $\dim_K(\ker_{V_1}(\partial - 1)) = 1$ . The map  $\tau - \lambda$  (where  $\lambda \neq 1$ ) is an isomorphism, therefore  $\text{ind}_{V_1}(\partial - \lambda) = 0$ .  $\square$

**Theorem 3.6** *Let  $a \in \mathbb{I}_1$ ,  $\cdot a : \mathbb{I}_1 \rightarrow \mathbb{I}_1$ ,  $b \mapsto ba$ ,  $a \cdot : \mathbb{I}_1 \rightarrow \mathbb{I}_1$ ,  $b \mapsto ab$ , and  $l_{\mathbb{I}_1}$  be the length function on the set of (left or right)  $\mathbb{I}_1$ -modules. Then*

1. (a)  $l_{\mathbb{I}_1}(\ker(\cdot a)) < \infty$  iff  $l_{\mathbb{I}_1}(\text{coker}(\cdot a)) < \infty$  iff  $a \notin F$ .  
(b)  $l_{\mathbb{I}_1}(\text{im}(\cdot a)) < \infty$  iff  $a \in F$ .
2. (a)  $l_{\mathbb{I}_1}(\ker(a \cdot)) < \infty$  iff  $l_{\mathbb{I}_1}(\text{coker}(a \cdot)) < \infty$  iff  $a \notin F$ .  
(b)  $l_{\mathbb{I}_1}(\text{im}(a \cdot)) < \infty$  iff  $a \in F$ .

*Proof.* 1(a). Applying the Snake Lemma to the commutative diagram of  $\mathbb{I}_1$ -modules

$$\begin{array}{ccccccc} 0 & \longrightarrow & F & \longrightarrow & \mathbb{I}_1 & \longrightarrow & B_1 \longrightarrow 0 \\ & & \downarrow \cdot a & & \downarrow \cdot a & & \downarrow \cdot \bar{a} \\ 0 & \longrightarrow & F & \longrightarrow & \mathbb{I}_1 & \longrightarrow & B_1 \longrightarrow 0 \end{array}$$

(where  $\bar{a} := a + F \in B_1$ ) yields the long exact sequence of  $\mathbb{I}_1$ -modules

$$0 \rightarrow \ker_F(\cdot a) \rightarrow \ker_{\mathbb{I}_1}(\cdot a) \rightarrow \ker_{B_1}(\cdot \bar{a}) \rightarrow \text{coker}_F(\cdot a) \rightarrow \text{coker}_{\mathbb{I}_1}(\cdot a) \rightarrow \text{coker}_{B_1}(\cdot \bar{a}) \rightarrow 0. \quad (15)$$

If  $a \in F$  then  $l_{\mathbb{I}_1}(\ker_F(\cdot a)) = \infty$  and  $l_{\mathbb{I}_1}(\text{coker}_{B_1}(\cdot \bar{a})) = l_{\mathbb{I}_1}(B_1) = \infty$  since  $\bar{a} = 0$ . Therefore,  $l_{\mathbb{I}_1}(\ker_{\mathbb{I}_1}(\cdot a)) = \infty$  and  $l_{\mathbb{I}_1}(\text{coker}_{\mathbb{I}_1}(\cdot a)) = \infty$ , by (15). To finish the proof it suffices to show that  $l_{\mathbb{I}_1}(\ker_{\mathbb{I}_1}(\cdot a)) < \infty$  and  $l_{\mathbb{I}_1}(\text{coker}_{\mathbb{I}_1}(\cdot a)) < \infty$  provided  $a \notin F$ . By (15), it suffices to show that the  $\mathbb{I}_1$ -modules  $\ker_F(\cdot a)$ ,  $\text{coker}_F(\cdot a)$ ,  $\ker_{B_1}(\cdot \bar{a})$  and  $\text{coker}_{B_1}(\cdot \bar{a})$  have finite length. Since  $\bar{a} \neq 0$ ,  $\ker_{B_1}(\cdot \bar{a}) = 0$  and  $l_{\mathbb{I}_1}(\text{coker}_{B_1}(\cdot \bar{a})) = l_{B_1}(\text{coker}_{B_1}(\cdot \bar{a})) < \infty$  since the algebra  $B_1$  is a localization

of the Weyl algebra  $A_1$  for which the analogous property holds (i.e.,  $l_{A_1}(A_1/A_1u) < \infty$  for all nonzero elements  $u \in A_1$ ). Notice that

$$F = \mathbb{I}_1 E_{00} \mathbb{I}_1 = \mathbb{I}_1 E_{00} E_{00} \mathbb{I}_1 = E_{\mathbb{N},0} E_{0,\mathbb{N}} = E_{\mathbb{N},0} \otimes E_{0,\mathbb{N}}$$

where  $\mathbb{I}_1 E_{\mathbb{N},0} = \mathbb{I}_1 E_{00} = \bigoplus_{i \in \mathbb{N}} K E_{i0} \simeq \mathbb{I}_1 K[x]$ ,  $E_{00} \mapsto 1$ , and  $(E_{0,\mathbb{N}})_{\mathbb{I}_1} = E_{00} \mathbb{I}_1 = \bigoplus_{i \in \mathbb{N}} K E_{0i} \simeq K[\partial]_{\mathbb{I}_1}$ ,  $E_{00} \mapsto 1$  where  $K[\partial]_{\mathbb{I}_1} \simeq \mathbb{I}_1 / \int \mathbb{I}_1$  is a simple right  $\mathbb{I}_1$ -module. By Theorem 3.1.(2a), the linear map  $a_{K[\partial]} : K[\partial] \rightarrow K[\partial]$ ,  $v \mapsto av$ , has finite dimensional kernel and cokernel since the module  $K[\partial]_{\mathbb{I}_1}$  is simple. Since

$$\ker_F(\cdot a) = E_{\mathbb{N},0} \otimes \ker(a_{K[\partial]}), \quad \text{coker}_F(\cdot a) \simeq E_{\mathbb{N},0} \otimes \text{coker}(a_{K[\partial]}), \quad (16)$$

and the  $\mathbb{I}_1$ -module  $E_{\mathbb{N},0} \simeq K[x]$  is simple, we see that

$$l_{\mathbb{I}_1}(\ker_F(\cdot a)) = \dim_K(\ker(a_{K[\partial]})) < \infty, \quad l_{\mathbb{I}_1}(\text{coker}_F(\cdot a)) = \dim_K(\text{coker}(a_{K[\partial]})) < \infty. \quad (17)$$

The proof of statement 1(a) is complete.

1(b). ( $\Rightarrow$ ) Note that  $l(\mathbb{I}_1 \mathbb{I}_1) = \infty$ . So, if  $l_{\mathbb{I}_1}(\text{im}(\cdot a)) < \infty$  then necessarily  $a \in F$ , by statement 1(a).

( $\Leftarrow$ ) If  $a \in F$  then, clearly  $l_{\mathbb{I}_1}(\text{im}(\cdot a)) < \infty$ .

2. Statement 2 follows from statement 1 by applying the involution  $*$  of the algebra  $\mathbb{I}_1$  and using the equality  $F^* = F$ .  $\square$

### The left and right length indices.

*Definition.* For each element  $a \in \mathbb{I}_1 \setminus F$ , we define its left and right *length index* respectively as follows (by Theorem 3.6)

$$\text{l.ind}(a) = l_{\mathbb{I}_1}(\ker(\cdot a)) - l_{\mathbb{I}_1}(\text{coker}(\cdot a)), \quad \text{r.ind}(a) = l_{\mathbb{I}_1}(\ker(a \cdot)) - l_{\mathbb{I}_1}(\text{coker}(a \cdot)).$$

The set  $\mathbb{I}_1 \setminus F$  is a multiplicative monoid. For all elements  $a, b \in \mathbb{I}_1 \setminus F$ ,

$$\text{l.ind}(ab) = \text{l.ind}(a) + \text{l.ind}(b), \quad \text{r.ind}(ab) = \text{r.ind}(a) + \text{r.ind}(b). \quad (18)$$

These equalities follow from (9). Therefore, the indices  $\text{l.ind}, \text{r.ind} : \mathbb{I}_1 \setminus F \rightarrow \mathbb{Z}$  are monoid homomorphisms. The next lemma show that the left and right indices are invariant under addition of compact operator.

**Lemma 3.7** *Let  $a \in \mathbb{I}_1 \setminus F$  and  $f \in F$ . Then  $\text{l.ind}(a + f) = \text{l.ind}(a)$  and  $\text{r.ind}(a + f) = \text{r.ind}(a)$ .*

*Proof.* Note that  $\partial^i f = 0$  for all  $i \gg 0$ , and  $\partial^i, \partial^i a \notin F$ . Then, by (18),

$$\text{l.ind}(\partial^i) + \text{l.ind}(a) = \text{l.ind}(\partial^i a) = \text{l.ind}(\partial^i(a + f)) = \text{l.ind}(\partial^i) + \text{l.ind}(a + f),$$

hence  $\text{l.ind}(a) = \text{l.ind}(a + f)$ . Replacing  $\text{l.ind}$  by  $\text{r.ind}$  in the argument above, the second equality follows.  $\square$

## 4 The algebra $\mathbb{I}_1$ is a coherent algebra

In this section, we prove that the algebra  $\mathbb{I}_1$  is a left and right coherent algebra (Theorem 4.4) and that every finitely generated left (or right) ideal of the algebra  $\mathbb{I}_1$  is generated by two elements (Theorem 4.5).

Let  $V$  be a vector space. A linear map  $\varphi : V \rightarrow V$  is called a *locally nilpotent* map if  $V = \bigcup_{i \geq 1} \ker(\varphi^i)$ , i.e., for each element  $v \in V$ ,  $\varphi^i v = 0$  for some  $i = i(v)$ .

**Lemma 4.1** *Let  $a, b \in \mathbb{I}_1$ . Let  $a \cdot$  and  $\cdot b$  be the left and right multiplication maps in  $\mathbb{I}_1$  by the elements  $a$  and  $b$  respectively.*

1. There are short exact sequences of left  $\mathbb{I}_1$ -modules:

$$(a) \ 0 \rightarrow \ker_F(\cdot b) \rightarrow \ker_{\mathbb{I}_1}(\cdot b) \rightarrow \ker_{B_1}(\cdot b) \rightarrow 0,$$

$$(b) \ 0 \rightarrow \operatorname{coker}_F(\cdot b) \rightarrow \operatorname{coker}_{\mathbb{I}_1}(\cdot b) \rightarrow \operatorname{coker}_{B_1}(\cdot b) \rightarrow 0.$$

2. There are long exact sequences of vector spaces:

$$(a) \ 0 \rightarrow \ker_{\ker_F(\cdot b)}(a \cdot) \rightarrow \ker_{\ker_{\mathbb{I}_1}(\cdot b)}(a \cdot) \rightarrow \ker_{\ker_{B_1}(\cdot b)}(a \cdot) \rightarrow \operatorname{coker}_{\ker_F(\cdot b)}(a \cdot) \rightarrow$$

$$\operatorname{coker}_{\ker_{\mathbb{I}_1}(\cdot b)}(a \cdot) \rightarrow \operatorname{coker}_{\ker_{B_1}(\cdot b)}(a \cdot) \rightarrow 0,$$

$$(b) \ 0 \rightarrow \ker_{\operatorname{coker}_F(\cdot b)}(a \cdot) \rightarrow \ker_{\operatorname{coker}_{\mathbb{I}_1}(\cdot b)}(a \cdot) \rightarrow \ker_{\operatorname{coker}_{B_1}(\cdot b)}(a \cdot) \rightarrow \operatorname{coker}_{\operatorname{coker}_F(\cdot b)}(a \cdot) \rightarrow$$

$$\operatorname{coker}_{\operatorname{coker}_{\mathbb{I}_1}(\cdot b)}(a \cdot) \rightarrow \operatorname{coker}_{\operatorname{coker}_{B_1}(\cdot b)}(a \cdot) \rightarrow 0.$$

*Proof.* 1. By (15), we have the long exact sequence of left  $\mathbb{I}_1$ -modules:

$$0 \rightarrow \ker_F(\cdot b) \rightarrow \ker_{\mathbb{I}_1}(\cdot b) \rightarrow \ker_{B_1}(\cdot b) \xrightarrow{\delta} \operatorname{coker}_F(\cdot b) \rightarrow \operatorname{coker}_{\mathbb{I}_1}(\cdot b) \rightarrow \operatorname{coker}_{B_1}(\cdot b) \rightarrow 0.$$

The map  $\delta$  is equal to zero since the element  $\partial$  acts as an invertible linear map on  $\ker_{B_1}(\cdot b)$  but its action on  $\operatorname{coker}_F(\cdot b)$  is a locally nilpotent map. So, the long exact sequence breaks down into two short exact sequences (as above).

2. Applying the Snake Lemma to the commutative diagrams of short exact sequences of vector spaces

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker_F(\cdot b) & \longrightarrow & \ker_{\mathbb{I}_1}(\cdot b) & \longrightarrow & \ker_{B_1}(\cdot b) \longrightarrow 0 \\ & & \downarrow a \cdot & & \downarrow a \cdot & & \downarrow a \cdot \\ 0 & \longrightarrow & \ker_F(\cdot b) & \longrightarrow & \ker_{\mathbb{I}_1}(\cdot b) & \longrightarrow & \ker_{B_1}(\cdot b) \longrightarrow 0, \\ \\ 0 & \longrightarrow & \operatorname{coker}_F(\cdot b) & \longrightarrow & \operatorname{coker}_{\mathbb{I}_1}(\cdot b) & \longrightarrow & \operatorname{coker}_{B_1}(\cdot b) \longrightarrow 0 \\ & & \downarrow a \cdot & & \downarrow a \cdot & & \downarrow a \cdot \\ 0 & \longrightarrow & \operatorname{coker}_F(\cdot b) & \longrightarrow & \operatorname{coker}_{\mathbb{I}_1}(\cdot b) & \longrightarrow & \operatorname{coker}_{B_1}(\cdot b) \longrightarrow 0. \end{array}$$

yields the long exact sequences of statement 2.  $\square$

**Theorem 4.2** *Let  $a \in \mathbb{I}_1$ . Then*

1.  $\ker_{\mathbb{I}_1}(\cdot a)$  and  $\operatorname{coker}_{\mathbb{I}_1}(\cdot a)$  are finitely generated left  $\mathbb{I}_1$ -modules.
2.  $\ker_{\mathbb{I}_1}(a \cdot)$  and  $\operatorname{coker}_{\mathbb{I}_1}(a \cdot)$  are finitely generated right  $\mathbb{I}_1$ -modules.

*Proof.* An algebra  $\mathbb{I}_1$  is self-dual, so it suffices to prove only statement 1. If  $a \notin F$  then statement 1 follows from Theorem 3.6.(1a). We may assume that  $a \in F$  and  $a \neq 0$ . By Lemma 4.1.(1), there are short exact sequences of left  $\mathbb{I}_1$ -modules

$$0 \rightarrow \ker_F(\cdot a) \rightarrow \ker_{\mathbb{I}_1}(\cdot a) \rightarrow B_1 \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \operatorname{coker}_F(\cdot a) \rightarrow \operatorname{coker}_{\mathbb{I}_1}(\cdot a) \rightarrow B_1 \rightarrow 0.$$

Fix elements  $u \in \ker_{\mathbb{I}_1}(\cdot a)$  and  $v \in \operatorname{coker}_{\mathbb{I}_1}(\cdot a)$  that are mapped to  $1 \in B_1$  by the second maps in the short exact sequences. Then  $u = 1 + f$  for some element  $f \in F$  and  $v = 1 + g + \mathbb{I}_1 a$  for some element  $g \in F$ . Since  $\mathbb{I}_1 u \subseteq \ker_{\mathbb{I}_1}(\cdot a)$  and  $l_{\mathbb{I}_1}(\mathbb{I}_1/\mathbb{I}_1 u) = l_{\mathbb{I}_1}(\operatorname{coker}_{\mathbb{I}_1}(\cdot u)) < \infty$  (Theorem 3.6.(2a) as  $u \notin F$ ), we see that  $l_{\mathbb{I}_1}(\ker_{\mathbb{I}_1}(\cdot a)/\mathbb{I}_1 u) < \infty$ , this means that the left  $\mathbb{I}_1$ -module  $\ker_{\mathbb{I}_1}(\cdot a)$  is finitely generated. Similarly, since  $\mathbb{I}_1 v \subseteq \operatorname{coker}_{\mathbb{I}_1}(\cdot a)$  and  $l_{\mathbb{I}_1}(\mathbb{I}_1/\mathbb{I}_1(1 + g)) = l_{\mathbb{I}_1}(\operatorname{coker}_{\mathbb{I}_1}(\cdot(1 + g))) < \infty$  (Theorem 3.6.(2a) as  $1 + g \notin F$ ), we see that

$$l_{\mathbb{I}_1}(\operatorname{coker}_{\mathbb{I}_1}(\cdot a)/\mathbb{I}_1 v) = l_{\mathbb{I}_1}(\mathbb{I}_1/(\mathbb{I}_1(1 + g) + \mathbb{I}_1 a)) \leq l_{\mathbb{I}_1}(\mathbb{I}_1/\mathbb{I}_1(1 + g)) < \infty.$$

This means that the left  $\mathbb{I}_1$ -module  $\operatorname{coker}_{\mathbb{I}_1}(\cdot a)$  is finitely generated.  $\square$

**Theorem 4.3** *The intersection of finitely many finitely generated left (resp. right) ideals of the algebra  $\mathbb{I}_1$  is again a finitely generated left (resp. right) ideal of  $\mathbb{I}_1$ .*

*Proof.* Since the algebra  $\mathbb{I}_1$  is self-dual it suffices to prove the statement for, say, left ideals, and only for two of them. Let  $I$  and  $J$  be finitely generated left ideals of the algebra  $\mathbb{I}_1$ . If one of them, say  $I$ , belongs to the ideal  $F$  then necessarily the left ideal  $I$  is a finitely generated semi-simple left  $\mathbb{I}_1$ -module, hence so is the intersection  $I \cap J$ . In particular,  $I \cap J$  is finitely generated.

We assume that neither  $I$  nor  $J$  belongs to  $F$ . Then their images  $\bar{I}$  and  $\bar{J}$  under the ring epimorphism  $\mathbb{I}_1 \rightarrow \mathbb{I}_1/F = B_1$  are nonzero left ideals of the ring  $B_1 = A_{1,x}$  which is the localization of the Weyl algebra  $A_1$  at the powers of the element  $x$ . Then  $\bar{I} \cap \bar{J} \neq 0$  (since  $\bar{I} \cap A_1 \neq 0$ ,  $\bar{J} \cap A_1 \neq 0$ , and the intersection of two nonzero left ideals in  $A_1$  is a nonzero left ideal). Take an element  $a \in \mathbb{I}_1$  such that  $0 \neq a + F \in \bar{I} \cap \bar{J}$ . Then  $a \notin F$  and  $\mathbb{I}_1 a \subseteq I \cap J$ . Since  $l_{\mathbb{I}_1}(\mathbb{I}_1/\mathbb{I}_1 a) < \infty$  (by Theorem 3.6.1(a)), the ideal  $I \cap J \neq 0$  is finitely generated.  $\square$

A finitely generated module is a *coherent* module if every finitely generated submodule is finitely presented. A ring  $R$  is a *left* (resp. *right*) *coherent ring* if the module  ${}_R R$  (resp.  $R_R$ ) is coherent. A ring  $R$  is a *left coherent ring* iff, for each element  $r \in R$ ,  $\ker_R(\cdot r)$  is a finitely generated left  $R$ -module and the intersection of two finitely generated left ideals is finitely generated, Proposition 13.3, [27]. Each left Noetherian ring is left coherent but not vice versa.

**Theorem 4.4** *The algebra  $\mathbb{I}_1$  is a left and right coherent algebra.*

*Proof.* The theorem follows from Theorem 4.2, Theorem 4.3 and Proposition 13.3, [27].  $\square$

**Theorem 4.5** 1. *Every finitely generated left (resp. right) ideal of the algebra  $\mathbb{I}_1$  is generated by two elements.*

2. *Let  $I$  be a left (resp. right) ideal of  $\mathbb{I}_1$ . Then*

- (a) *If  $I \not\subseteq F$  then the left (resp. right) ideal  $I$  is generated by two elements.*
- (b) *If  $I \subseteq F$  and  $I$  is a finitely generated left (resp. right) ideal then  $I$  is generated by a single element.*

*Proof.* The algebra  $\mathbb{I}_1$  is self-dual, so it suffices to prove the statements for left ideals.

1. Statement 1 follows from statement 2.

2(a). Since  $I \not\subseteq F$ , we can fix an element  $a \in I \setminus F$ . By Theorem 3.6.1(a), the factor module  $I/\mathbb{I}_1 a$  has finite length, and so is cyclic, by Proposition 4.6. Therefore,  ${}_1 I$  is generated by two elements.

2(b).  ${}_1 F = \bigoplus_{i \in \mathbb{N}} \mathbb{I}_1 e_{0i}$  is the direct sum of simple isomorphic  $\mathbb{I}_1$ -modules  $\mathbb{I}_1 e_{0i} = \bigoplus_{j \in \mathbb{N}} K e_{ji}$ . Then  $I \simeq \bigoplus_{i=0}^s \mathbb{I}_1 e_{0i} =: I'$  for some  $s$ . Clearly,  $I' = \mathbb{I}_1 \theta$  where  $\theta = e_{00} + e_{11} + \cdots + e_{ss}$  since  $\partial^s \theta = e_{0s}$  and so  $\mathbb{I}_1 e_{0s} \in \mathbb{I}_1 \theta$  and  $\mathbb{I}_1 \theta = \mathbb{I}_1 (e_{00} + e_{11} + \cdots + e_{s-1,s-1}) + \mathbb{I}_1 e_{0s}$ . Using a similar argument we see that

$$\mathbb{I}_1 \theta = \mathbb{I}_1 (e_{00} + e_{11} + \cdots + e_{s-2,s-2}) + \mathbb{I}_1 e_{0,s-1} + \mathbb{I}_1 e_{0s} = \cdots = \sum_{i=1}^s \mathbb{I}_1 e_{0j} = I'. \quad \square$$

**Proposition 4.6** *Every left or right  $\mathbb{I}_1$ -module of finite length is cyclic (i.e., generated by a single element).*

*Proof.* The algebra  $\mathbb{I}_1$  is self-dual, so it suffices to show that every left  $\mathbb{I}_1$ -module of finite length  $M$  is cyclic. We use induction on the length  $l = l_{\mathbb{I}_1}(M)$  of the module  $M$ . The case  $l = 1$  is trivial. So, let  $l > 1$ , and we assume that the statement holds for all  $l' < l$ . Fix a simple submodule  $U = \mathbb{I}_1 u \simeq \mathbb{I}_1/\mathfrak{a}$  of  $M$  where  $u \in M$  and  $\mathfrak{a} = \text{ann}_U(u)$ . Let  $V := M/U$ . Then  $l_{\mathbb{I}_1}(V) = l - 1$ , and, by induction, the  $\mathbb{I}_1$ -module  $V = \mathbb{I}_1 \bar{v} \simeq \mathbb{I}_1/\mathfrak{b}$  is cyclic where  $\bar{v} \in V$  and  $\mathfrak{b} = \text{ann}_V(\bar{v})$ . Fix an element  $v \in M$  such that  $\bar{v} = v + U$ . Let  $\mathfrak{c} = \mathfrak{a} \cap \mathfrak{b}$ . Then  $l_{\mathbb{I}_1}(\mathbb{I}_1/\mathfrak{c}) < \infty$  since  $\mathbb{I}_1/\mathfrak{c}$  can be seen as

a submodule of the finite length  $\mathbb{I}_1$ -module  $\mathbb{I}_1/\mathfrak{a} \oplus \mathbb{I}_1/\mathfrak{b}$ . Then  $\mathfrak{c} \neq 0$  since  $l_{\mathbb{I}_1}(\mathbb{I}_1) = \infty$ . Moreover,  $\mathfrak{c} \not\subseteq F$  since  $l_{\mathbb{I}_1}(\mathbb{I}_1/F) = l_{B_1}(B_1) = \infty$ . We claim that there exists an element  $a \in \mathbb{I}_1$  such that  $cau \neq 0$ . Suppose not, i.e.,  $\mathfrak{c}\mathbb{I}_1U = 0$ , we seek a contradiction. Then  $\mathfrak{c}\mathbb{I}_1 = F$  or  $\mathbb{I}_1$  since  $F$  and  $\mathbb{I}_1$  are the only nonzero ideals of the algebra  $\mathbb{I}_1$ , [12]. The first case is not possible since  $\mathfrak{c} \not\subseteq F$ . The second case is not possible since  $u \neq 0$ . This finishes the proof of the claim. Fix  $c \in \mathfrak{c}$  and  $a \in \mathbb{I}_1$  such that  $cau \neq 0$ . Then the element  $w = au + v$  is a generator for the  $\mathbb{I}_1$ -module  $M$ . Indeed,  $cw = cau \neq 0$ . Then,  $U = \mathbb{I}_1cau \subseteq \mathbb{I}_1w$ , and so  $v \in \mathbb{I}_1w$ . Therefore,  $M = \mathbb{I}_1w$ .  $\square$

## 5 Centralizers

The centralizers of non-scalar elements of the Weyl algebra  $A_1$  share many pleasant properties. Recall Amitsur's well-known theorem on the centralizer [1] which states that the centralizer  $\text{Cen}_{A_1}(a)$  of any non-scalar element  $a$  of the Weyl algebra  $A_1$  is a commutative algebra and a free  $K[a]$ -module of finite rank (see also Burchall and Chaundy [19]). In particular, the centralizer  $\text{Cen}_{A_1}(a)$  is a commutative finitely generated (hence Noetherian) algebra. It turns out that this result also holds for certain generalized Weyl algebra [4], [8], [9], [10] and some (quantum) algebras see [19], [20], [22], [16], [2], [24], [23]. Proposition 5.1 shows that the situation is completely different for the algebra  $\mathbb{I}_1$ . Theorem 5.7 presents in great detail the structure of the centralizers of the non-scalar elements of the algebra  $\mathbb{I}_1$ , and Corollary 5.8 answers the questions of when the centralizer  $\text{Cen}_{\mathbb{I}_1}(a)$  is a finitely generated  $K[a]$ -module where  $a \in \mathbb{I}_1$ , or is a finitely generated Noetherian algebra. Corollary 5.9 classifies the non-scalar elements of the algebra  $\mathbb{I}_1$  such that their centralizers are finitely generated algebras. The next proposition will be used in the proof of Theorem 5.7.

For an element  $a \in \mathbb{I}_1$ , let  $\text{Cen}_{\mathbb{I}_1}(a) := \{b \in \mathbb{I}_1 \mid ab = ba\}$  be its *centralizer* in  $\mathbb{I}_1$  and  $\text{Cen}_F(a) := F \cap \text{Cen}_{\mathbb{I}_1}(a)$ .

**Proposition 5.1** 1. Let  $\alpha \in K[H] \setminus K$ . Then  $\text{Cen}_{\mathbb{I}_1}(\alpha) = D_1 \oplus C_\alpha \oplus C_\alpha^*$  where the vector space  $C_\alpha := \bigoplus_{i \geq 1, j \geq 0} \{Ke_{i+j,j} \mid j+1 \text{ is a root of the polynomial } \tau^i(\alpha) - \alpha\}$ , and  $\text{Cen}_{\mathbb{I}_1}(\alpha)^* = \text{Cen}_{\mathbb{I}_1}(\alpha)$ . In general, the centralizer  $\text{Cen}_{\mathbb{I}_1}(\alpha)$  is a noncommutative, not left and not right Noetherian, not finitely generated algebra which is not a domain as the following example shows:  $\text{Cen}_{\mathbb{I}_1}((H - 3/2)^2) = D_1 \oplus Ke_{10} \oplus Ke_{01}$ ; but  $\text{Cen}_{\mathbb{I}_1}(H^k) = D_1$  is a commutative, not Noetherian, not finitely generated algebra which is not a domain for all  $k \geq 1$ ; and  $\text{Cen}_{\mathbb{I}_1}(H - 3/2) = D_1 \neq \text{Cen}_{\mathbb{I}_1}((H - 3/2)^2)$ .

2. Let  $a \in \mathbb{I}_1$ . Then  $\dim_K(\text{Cen}_F(a)) < \infty$  iff  $a \notin K[H] + F$ .

3.  $\text{Cen}_{\mathbb{I}_1}(\partial^i) = K[\partial]$  and  $\text{Cen}_{\mathbb{I}_1}(f^i) = K[f]$  for all  $i \geq 1$ .

4.  $\text{Cen}_{\mathbb{I}_1}(x^i) = K[x]$  for all  $i \geq 1$ .

*Proof.* 1. Recall that  $B_1 = \mathbb{I}_1/F$ . Notice that  $\text{Cen}_{B_1}(\alpha) = K[H]$ . This follows from the fact that the algebra of  $\tau^i$ -invariants  $K[H]^{\tau^i} := \{\beta \in K[H] \mid \tau^i(\beta) = \beta\}$  is equal to  $K$  for all integers  $0 \neq i \in \mathbb{Z}$ . Since  $\alpha^* = \alpha$ ,  $\text{Cen}_{\mathbb{I}_1}(\alpha)^* = \text{Cen}_{\mathbb{I}_1}(\alpha)$ . The element  $\alpha \in D_1$  is a homogeneous element of the  $\mathbb{Z}$ -graded algebra  $\mathbb{I}_1$ . Therefore, its centralizer is a homogeneous subalgebra of  $\mathbb{I}_1$ . Since  $K[H] \subseteq D_1 \subseteq \text{Cen}_{\mathbb{I}_1}(\alpha)$  and  $\text{Cen}_{B_1}(\alpha) = K[H]$ , we see that  $\text{Cen}_{\mathbb{I}_1}(\alpha) = D_1 \oplus \bigoplus_{i \geq 1} (C_i \oplus C_i^*)$  where  $C_i := F_{1,i} \cap \text{Cen}_{\mathbb{I}_1}(\alpha)$  and  $F_{1,i} := \bigoplus_{j \geq 0} Ke_{i+j,j}$ . Each direct summand  $Ke_{i+j,j}$  of the vector space  $F_{1,i}$  is a  $D_1$ -bimodule and  $\alpha \in D_1$ , hence  $C_i = \sum \{Ke_{i+j,j} \mid \alpha e_{i+j,j} = e_{i+j,j} \alpha\}$ . The equality in the brackets is equivalent to the equality  $\alpha(i+j+1) = \alpha(j+1)$ , i.e.,  $(\tau^i(\alpha) - \alpha)(j+1) = 0$ , i.e.,  $j+1$  is a root of the polynomial  $\tau^i(\alpha) - \alpha$ . Therefore,  $\text{Cen}_{\mathbb{I}_1}(\alpha) = D_1 \oplus C_\alpha \oplus C_\alpha^*$ . If  $\alpha = H^k$  then  $(\tau^i(H^k) - H^k)(j+1) = (i+j+1)^k - (j+1)^k > 0$ , and so  $\text{Cen}_{\mathbb{I}_1}(H^k) = D_1$  is a commutative, not Noetherian, not finitely generated algebra which is not a domain. If  $\alpha = (H - \frac{3}{2})^2$  then  $0 = (\tau^i(\alpha) - \alpha)(j+1) = i(2j+i-1)$ , and so  $C_\alpha = Ke_{10}$ . The centralizer  $\text{Cen}_{\mathbb{I}_1}(\alpha)$  is a noncommutative algebra (since  $He_{10} = 2e_{10} \neq e_{10} = e_{10}H$ ) and is not a domain (since  $e_{10}^2 = 0$ ). The factor algebra  $\mathcal{D} := D_1/(Ke_{00} + Ke_{11})$  is a commutative, not Noetherian, not



finitely generated algebra. Since the algebra  $\mathcal{D}$  is the factor algebra algebra of the algebra  $\text{Cen}_{\mathbb{I}_1}(\alpha)$  modulo the ideal  $\bigoplus_{i,j=0}^1 Ke_{ij}$ , the algebra  $\text{Cen}_{\mathbb{I}_1}(\alpha)$  is not a finitely generated algebra which is neither left nor right Noetherian. If  $\alpha = H - 3/2$  then the equation  $0 = (\tau^i(\alpha) - \alpha)(j+1) = i$  has no solution since  $i \geq 1$ . Therefore,  $\text{Cen}_{\mathbb{I}_1}(H - 3/2) = D_1 \neq \text{Cen}_{\mathbb{I}_1}((H - 3/2)^2)$ . Notice that, for the Weyl algebra  $A_1$ ,  $\text{Cen}_{A_1}(p(a)) = \text{Cen}_{A_1}(a)$  for all elements  $a \in A_1$  and  $p(t) \in K[t]$ , [20].

2. Suppose that  $a \in K[H] + F$ , i.e.,  $a = \alpha + \sum_{i,j=0}^n \lambda_{ij} e_{ij}$  for some  $n$  where  $\alpha \in K[H]$  and  $\lambda_{ij} \in K$ . Then  $\dim_K(\text{Cen}_F(a)) = \infty$  since  $\bigoplus_{i=n+1}^\infty Ke_{ii} \subseteq \text{Cen}_F(a)$ . It remains to show that if  $a \notin K[H] + F$  then  $\dim_K(\text{Cen}_F(a)) < \infty$ . By (7),  $a = \sum_{i=m}^n a_i v_i + a_F$  where  $a_F := \sum_{i,j=0}^l \lambda_{ij} e_{ij}$  for some elements  $a_i \in K[H]$ ,  $a_n \neq 0$ ,  $a_m \neq 0$ , and  $\lambda_{ij} \in K$ . Fix a natural number, say  $N$ , such that  $N > l$ ,  $Ke_{s+n,t} \ni a_n(H)v_n \cdot e_{st} = a_n(s+n+1)e_{s+n,t} \neq 0$  and  $Ke_{t,s-m} \ni e_{ts} \cdot a_m(H)v_m = a_m(s+1)e_{t,s-m} \neq 0$  for all  $s \geq N$  and  $t \in \mathbb{N}$ .

*Claim 1.* If  $m < 0$  then  $\text{Cen}_F(a) \subseteq \bigoplus_{j=0}^{N-1} E_{\mathbb{N},j}$  where  $E_{\mathbb{N},j} := \bigoplus_{i \in \mathbb{N}} Ke_{ij}$ .

Suppose that this is not the case then there exists an element  $c = \sum c_{ij} e_{ij}$  where  $c_{ij} \in K$  such that  $t := \max\{j \mid c_{ij} \neq 0 \text{ for some } i \in \mathbb{N}\} \geq N$ , we seek a contradiction. Fix an element  $s \in \mathbb{N}$  such that  $c_{st} \neq 0$ . Then  $ce_{t+|m|,t+|m|} = 0$  and  $a_F e_{t+|m|,t+|m|} = 0$ , and so

$$\begin{aligned} 0 &= e_{ss} \cdot 0 \cdot e_{t+|m|,t+|m|} = e_{ss}[c, a]e_{t+|m|,t+|m|} = e_{ss}cae_{t+|m|,t+|m|} - e_{ss}ace_{t+|m|,t+|m|} \\ &= \left(\sum c_{sj} e_{sj}\right) \cdot \left(\sum_{i=m}^n a_i v_i\right) \cdot e_{t+|m|,t+|m|} = c_{st}(e_{st}a_m v_m)e_{t+|m|,t+|m|} = c_{st}a_m(t+1)e_{s,t+|m|} \\ &\neq 0, \end{aligned}$$

since  $e_{st}a_m v_m = a_m(t+1)e_{s,t+|m|} \neq 0$ , by the choice of  $N$ , where  $a_m(t+1)$  is the value of the polynomial  $a_m(H)$  at  $H = t+1$ . This contradiction,  $0 \neq 0$ , finishes the proof of Claim 1. By applying the involution  $*$  of the algebra  $\mathbb{I}_1$  to Claim 1 we obtain the following statement.

*Claim 2.* If  $n > 0$  then  $\text{Cen}_F(a) \subseteq \bigoplus_{i=0}^{N-1} E_{i,\mathbb{N}}$  where  $E_{i,\mathbb{N}} := \bigoplus_{j \in \mathbb{N}} Ke_{ij}$ .

If  $m < 0$  and  $n > 0$  then, by Claims 1 and 2,

$$\text{Cen}_F(a) \subseteq \left(\bigoplus_{j=0}^{N-1} E_{\mathbb{N},j}\right) \cap \left(\bigoplus_{i=0}^{N-1} E_{i,\mathbb{N}}\right) = \bigoplus_{i,j=0}^{N-1} Ke_{ij},$$

and so  $\dim_K(\text{Cen}_F(a)) < \infty$ .

In view of the involution  $*$  of the algebra  $\mathbb{I}_1$ , to finish the proof of statement 2 it suffices to consider the case where  $m < 0$  and  $m \leq n \leq 0$ . By Claim 1,  $\text{Cen}_F(a) \subseteq V := \bigoplus_{j=0}^{N-1} E_{\mathbb{N},j}$ . Suppose that  $\dim_K(\text{Cen}_F(a)) = \infty$  (we seek a contradiction). Fix a natural number  $M$  such that  $M > N + |m|$ . The subspace  $U := \bigoplus_{j=0}^{N-1} \bigoplus_{i \geq M} Ke_{ij}$  of  $V$  has finite codimension, i.e.,  $\dim_K(V/U) < \infty$ . Therefore,  $I := \text{Cen}_F(a) \cap U \neq 0$  since  $\dim_K(\text{Cen}_F(a)) = \infty$ . Choose a nonzero element, say  $u = \sum u_{ij} e_{ij}$  (where  $u_{ij} \in K$ ), of the intersection  $I$ . Let  $p := \min\{i \in \mathbb{N} \mid u_{ij} \neq 0 \text{ for some } j \in \mathbb{N}\}$ . Then  $p \geq M$ , by the choice of the element  $u$ . Fix an element  $q \in \mathbb{N}$  such that  $u_{pq} \neq 0$ . Then  $e_{p-|m|,p-|m|}a_F = 0$  (since  $p - |m| \geq M - |m| > N + |m| - |m| = N > l$ ) and  $e_{p-|m|,p-|m|}u = 0$  (by the choice of  $p$  and since  $m < 0$ ), and so

$$\begin{aligned} 0 &= e_{p-|m|,p-|m|} \cdot 0 \cdot e_{qq} = e_{p-|m|,p-|m|}[a, u]e_{qq} = e_{p-|m|,p-|m|}aue_{qq} - e_{p-|m|,p-|m|}uae_{qq} \\ &= e_{p-|m|,p-|m|} \cdot \left(\sum_{j=m}^n a_j v_j\right) \cdot \left(\sum_{i \geq p} u_{iq} e_{iq}\right) = (e_{p-|m|,p-|m|}a_m v_m) \cdot u_{pq} e_{pq} \\ &= a_m(p - |m| + 1)u_{pq} e_{p-|m|,q} \neq 0, \end{aligned}$$

since  $e_{p-|m|,p-|m|}a_m v_m \neq 0$ , by the choice of  $N$ , and  $p - |m| \geq M - |m| > N + |m| - |m| = N$ . The contradiction,  $0 \neq 0$ , proves statement 2.

3. Let us prove the first equality then the second one becomes obvious:

$$\text{Cen}_{\mathbb{I}_1}\left(\int^i\right) = \text{Cen}_{\mathbb{I}_1}((\partial^i)^*) = (\text{Cen}_{\mathbb{I}_1}(\partial^i))^* = K[\partial]^* = K\left[\int\right].$$

Notice that  $\text{Cen}_F(\partial^i) = 0$ ,  $K[\partial] \subseteq C := \text{Cen}_{\mathbb{I}_1}(\partial^i)$  and  $\text{Cen}_{B_1}(\partial^i) = K[\partial, \partial^{-1}]$ , and therefore  $C \cap (F + K[\partial]) = K[\partial]$  and  $C \subseteq F + K[\partial] + K[f]$ . To finish the proof it suffices to show that  $C \subseteq F + K[\partial]$ . If  $C \not\subseteq F + K[\partial]$  then there exists an element  $c = \int + f$  for some element  $f \in F$  (this follows from the fact that  $\partial^i \int^i = 1$  and  $K[\partial] \subseteq C$ ), we seek a contradiction. Necessarily,  $f \neq 0$  as  $\int \notin C$ . Then  $\partial c = 1 + \partial f \in C \cap (F + K[\partial]) = K[\partial]$ , and so  $\partial f = 0$ , i.e.,  $f = \sum_{j \geq 0} \lambda_j e_{0j}$  where  $\lambda_j \in K$  and not all  $\lambda_j$  are equal to zero. Similarly,  $c\partial = 1 - e_{00} + f\partial \in C \cap (F + K[\partial]) = K[\partial]$ , and so  $e_{00} = f\partial = \sum_{j \geq 0} \lambda_j e_{0,j+1}$ , a contradiction. Then,  $C = K[\partial]$ .

4. Notice that  $\text{Cen}_F(x^i) = 0$ ,  $K[x] \subseteq C := \text{Cen}_{\mathbb{I}_1}(x^i)$  and  $\text{Cen}_{B_1}(x^i) = K[x]$ . The last equality implies that  $C \subseteq F + K[x]$ , then the first two give the result:  $C = \text{Cen}_F(x^i) + K[x] = K[x]$ .  $\square$

The following trivial lemma is a reason why the centralizers of elements may share ‘exotic’ properties.

**Lemma 5.2** *Let  $r$  be an element of a ring  $R$  and  $\mathfrak{a} := \text{l.ann}_R(r) \cap \text{r.ann}_R(r)$ . Then  $\mathfrak{a}R\mathfrak{a} \subseteq \text{Cen}_R(r)$ .*

The ideal  $F = \bigoplus_{i,j \in \mathbb{N}} Ke_{ij} = \bigoplus_{i,j \in \mathbb{N}} KE_{ij} \simeq M_\infty(K)$  of the algebra  $\mathbb{I}_1$  admits the *trace* (linear) map

$$\text{tr} : F \rightarrow K, \quad \sum \lambda_{ij} e_{ij} = \sum \mu_{ij} E_{ij} \mapsto \sum \lambda_{ii} = \sum \mu_{ii}$$

since  $E_{ij} = \frac{i!}{j!} e_{ij}$ . Clearly, for all elements  $\alpha \in K[H]$ ,

$$\text{tr}([\alpha \partial^i, F]) = \text{tr}([\alpha \int^i, F]) = \text{tr}([F, F]) = 0, \quad i \geq 0, \quad (19)$$

since  $[\alpha(H)v_i, e_{st}] = \alpha(i+s+1)e_{i+s,t} - \alpha(t+1)e_{s,t-i}$  where  $[a, F] := \{[a, f] := af - fa \mid f \in F\}$  for an element  $a \in \mathbb{I}_1$ , and  $[F, F]$  is the linear subspace of  $F$  generated by all the commutators  $[f, g]$  where  $f, g \in F$ . Therefore, by (7) and (19),

$$\text{tr}([\mathbb{I}_1, F]) = 0, \quad (20)$$

where  $[\mathbb{I}_1, F]$  is the subspace of  $F$  generated by all the commutators  $[a, f]$  where  $a \in \mathbb{I}_1$  and  $f \in F$ .

**Lemma 5.3** *For all positive integers  $i$  and  $j$ , and for all elements  $f, g \in F$ ,  $[\partial^i + f, \int^j + g] \neq 0$ .*

*Proof.* Suppose that  $[\partial^i + f, \int^j + g] = 0$ , we seek a contradiction. Then  $[(\partial^i + f)^j, (\int^j + g)^i] = 0$ , so we may assume that  $i = j$  since  $(\partial^i + f)^j = \partial^{ij} + f'$  and  $(\int^j + g)^i = \int^{ij} + g'$  for some elements  $f', g' \in F$ . Then the equality  $[\partial^i + f, \int^i + g] = 0$  implies the equalities (see (3))

$$e_{00} + e_{11} + \cdots + e_{i-1,i-1} = [\partial^i, \int^i] = -[\partial^i, g] - [f, \int^i] - [f, g].$$

Applying the trace map and using (19) we get a contradiction,  $i = 0$ .  $\square$

The algebra  $B_1$  is a subalgebra of the algebra  $\mathcal{B}_1 = K(H)[\partial, \partial^{-1}; \tau]$  which is the (two-sided) localization of the algebra  $B_1$  at the (left and right) denominator set  $K[H] \setminus \{0\}$ . A polynomial  $f = \lambda_n H^n + \lambda_{n-1} H^{n-1} + \cdots + \lambda_0 \in K[H]$  of degree  $n$  is called a *monic* polynomial if the leading coefficient  $\lambda_n$  of  $f$  is 1. A rational function  $h \in K(H)$  is called a *monic* rational function if  $h = f/g$  for some monic polynomials  $f, g$ . A homogeneous element  $u = \alpha \partial^i$  of  $\mathcal{B}_1$  is called *monic* iff  $\alpha$  is a monic rational function. We can extend in the obvious way the concept of the degree of a polynomial to the field of rational functions by setting,  $\deg_H(h) = \deg_H(f) - \deg_H(g)$ , for  $h = f/g \in K(H)$ . If  $h_1, h_2 \in K(H)$  then  $\deg_H(h_1 h_2) = \deg_H(h_1) + \deg_H(h_2)$ , and  $\deg_H(h_1 + h_2) \leq \max\{\deg_H(h_1), \deg_H(h_2)\}$ . We denote by  $\text{sign}(n)$  and by  $|n|$  the *sign* and the *absolute value* of  $n \in \mathbb{Z} \setminus \{0\}$ , respectively.

**Proposition 5.4** (Proposition 2.1, [9])

1. Let  $u = \alpha \partial^n$  be a monic element of  $\mathcal{B}_1$  with  $n \in \mathbb{Z} \setminus \{0\}$ . The centralizer  $\text{Cen}_{\mathcal{B}_1}(u) = K[v, v^{-1}]$  is a Laurent polynomial ring in a uniquely defined variable  $v = \beta \partial^{\text{sign}(n)s}$  where  $s$  is the least positive divisor of  $n$  for which there exists a monic element  $\beta = \beta_s \in K(H)$ , (necessarily unique) such that

$$\begin{aligned} \beta \tau^s(\beta) \tau^{2s}(\beta) \cdots \tau^{(n/s-1)s}(\beta) &= \alpha, \text{ if } n > 0, \\ \beta \tau^{-s}(\beta) \tau^{-2s}(\beta) \cdots \tau^{-(|n|/s-1)s}(\beta) &= \alpha, \text{ if } n < 0. \end{aligned}$$

2. Let  $u \in K(H) \setminus K$ . Then  $\text{Cen}_{\mathcal{B}_1}(u) = K(H)$ .

The algebra  $B_1 = K[H][\partial, \partial^{-1}; \tau] = \bigoplus_{i \in \mathbb{Z}} K[H] \partial^i$  is a  $\mathbb{Z}$ -graded algebra where  $K[H] \partial^i$  is the  $i$ 'th graded component. Each nonzero element  $b \in B_1$  is a unique finite sum  $b = \sum \beta_i \partial^i$  where  $\beta_i \in K[H]$ . Let

$$\pi'_+(b) := \max\{i \in \mathbb{Z} \mid \beta_i \neq 0\} \text{ and } \pi'_-(b) := \min\{i \in \mathbb{Z} \mid \beta_i \neq 0\} \quad (21)$$

For all elements  $a, b \in B_1 \setminus \{0\}$  and  $\alpha \in K[H] \setminus \{0\}$ ,  $\pi'_\pm(ab) = \pi'_\pm(a) + \pi'_\pm(b)$  and  $\pi'_\pm(\alpha a) = \pi'_\pm(a)$ . The element  $\beta_n \partial^n$  where  $n = \pi'_+(a)$  is called the *leading term* of the element  $b$ , and the element  $\beta_m \partial^m$  where  $m = \pi'_-(a)$  is called the *least term* of  $b$ .

**Corollary 5.5** 1. Let  $b \in B_1 \setminus K$ . Suppose that  $n = \pi'_+(b) > 0$  and  $g_1, g_2 \in \text{Cen}_{B_1}(b)$  be such that  $m = \pi'_+(g_1) = \pi'_+(g_2)$ . Let  $\beta_1 \partial^m$  and  $\beta_2 \partial^m$  be the leading terms of the elements  $g_1$  and  $g_2$  respectively where  $\beta_1, \beta_2 \in K[H]$ . Then  $K\beta_1 = K\beta_2$ .

2. Let  $b = \alpha \partial^n$  where  $\alpha \in K[H] \setminus K$  and  $n \in \mathbb{Z} \setminus \{0\}$ . Then  $\text{Cen}_{B_1}(b) \cap B_{1, -\text{sign}(n)} = K$  where  $B_{1, -} := K[H][\partial^{-1}; \tau^{-1}] \subseteq B_1$  and  $B_{1, +} := K[H][\partial; \tau] \subseteq B_1$ .

*Proof.* 1. Let  $\beta \partial^n$  be the leading term of the element  $b$ . The elements  $b$  and  $g_1$  (respectively,  $b$  and  $g_2$ ) commute then so do their leading terms. Then the result follows from Proposition 5.4.(1) since  $\text{Cen}_{B_1}(\beta \partial^n) \subseteq \text{Cen}_{\mathcal{B}_1}(\beta \partial^n)$ .

2. Without loss of generality we may assume that  $\alpha$  is a monic since  $\text{Cen}_{B_1}(\lambda b) = \text{Cen}_{B_1}(b)$  for all  $\lambda \in K^*$ . In view of the  $(\pm)$ -symmetry we may assume that  $n > 0$ . By Proposition 5.4,  $\text{Cen}_{\mathcal{B}_1}(b) = K[v, v^{-1}]$  for some element  $v = \beta \partial^s$  where  $s > 0$ ,  $s|n$ ,  $\beta \in K(H)$ , and  $v^{\frac{n}{s}} = (\beta \partial^s)^{\frac{n}{s}} = \beta \tau^s(\beta) \cdots \tau^{(n/s-1)s}(\beta) \partial^n = \alpha \partial^n = b$ . Clearly,

$$1 \leq \deg_H(\alpha) = \deg_H\left(\prod_{j=0}^{n/s-1} \tau^{js}(\beta)\right) = \sum_{j=0}^{n/s-1} \deg_H(\tau^{js}(\beta)) = \sum_{j=0}^{n/s-1} \deg_H(\beta) = \frac{n}{s} \deg_H(\beta),$$

and so  $\deg_H(\beta) > 0$ . Clearly,  $\text{Cen}_{B_1}(b) \cap B_{1, -} \subseteq \text{Cen}_{\mathcal{B}_1}(b) \cap B_{1, -} \subseteq K[v^{-1}] \cap B_1 = \bigoplus_{i \in \mathbb{N}} (K v^{-i} \cap B_1)$ . For all integers  $i \geq 1$ ,

$$v^{-i} = (\partial^{-s} \beta^{-1})^i = \partial^{-si} \beta^{-1} \tau^s(\beta^{-1}) \cdots \tau^{s(i-1)}(\beta^{-1}) \notin B_1$$

since  $\deg_H(\beta^{-1} \tau^s(\beta^{-1}) \cdots \tau^{s(i-1)}(\beta^{-1})) = s \deg_H(\beta^{-1}) = -s \deg_H(\beta) < 0$ . Therefore,  $\text{Cen}_{B_1}(b) \cap B_{1, -} = K$ .  $\square$

For each element  $a \in \mathbb{I}_1 \setminus F$ , at least one of the elements  $b_i \in K[H]$  in (7) is nonzero. Let

$$\pi_+(a) := \max\{i \in \mathbb{Z} \mid b_i \neq 0\}, \quad \pi_-(a) := \min\{i \in \mathbb{Z} \mid b_i \neq 0\}. \quad (22)$$

For the element  $a \in \mathbb{I}_1 \setminus F$ , the summands  $l_+(a) := b_m v_m$  and  $l_-(a) := b_n v_n$  where  $m := \pi_+(a)$  and  $n := \pi_-(a)$  are called the *largest* and *least terms* of  $a$  modulo  $F$  respectively. It is a useful observation that if  $[a, b] = 0$  then

$$[l_+(a), l_+(b)], [l_-(a), l_-(b)] \in F. \quad (23)$$

For an algebra  $A$ , the subspace  $[A, A]$  of  $A$  generated by all the commutators  $[a, b] := ab - ba$  where  $a, b \in A$  is called the *commutant* of the algebra  $A$ . By (7),  $\mathbb{I}_1 = \bigoplus_{i \in \mathbb{Z}} K[H]v_i \bigoplus \bigoplus_{s, t \in \mathbb{N}} Ke_{st}$ . Let

$$\xi : \mathbb{I}_1 \rightarrow \mathbb{I}_1 \quad (24)$$

be the projection onto the direct summand  $\bigoplus_{i \in \mathbb{N}} Ke_{ii}$  of  $\mathbb{I}_1$ .

We identify each linear map  $b \in \mathbb{I}_1 \subseteq \text{End}_K(K[x])$  with its  $\mathbb{N} \times \mathbb{N}$  matrix with respect to the  $K$ -basis  $\{x^{[s]} := \frac{x^s}{s!} \mid s \in \mathbb{N}\}$  of the vector space  $K[x]$ . For each natural number  $d \in \mathbb{N}$ , let  $e_d := e_{00} + e_{11} + \dots + e_{dd}$  and  $e'_d := 1 - e_d$ . Then  $1 = e_d + e'_d$  is the sum of two orthogonal idempotents in the algebra  $\mathbb{I}_1 \subseteq \text{End}_K(K[x])$ . Therefore,  $K[x] = \text{im}_{K[x]}(e_d) \bigoplus \text{im}_{K[x]}(e'_d) = K[x]_{\leq d} \bigoplus K[x]_{> d}$  and, for each element  $b \in \mathbb{I}_1$ ,

$$b = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad b_{11} = e_d b e_d, \quad b_{12} := e_d b e'_d, \quad b_{21} = e'_d b e_d, \quad b_{22} := e'_d b e'_d. \quad (25)$$

Notice that  $b_{11} \in F_{\leq d} := \bigoplus_{i,j=0}^d Ke_{ij}$ ,  $b_{12} \in \bigoplus_{i=0}^d \bigoplus_{j>d} Ke_{ij}$ ,  $b_{21} \in \bigoplus_{i>d} \bigoplus_{j=0}^d Ke_{ij}$  and  $b_{22} \in \bigoplus_{i,j>d} Ke_{ij}$ . For each natural number  $d \in \mathbb{N}$ , consider the algebra  $e'_d \mathbb{I}_1 e'_d \subseteq \mathbb{I}_1$  where  $e'_d$  is its identity element. For each element  $b \in e'_d \mathbb{I}_1 e'_d$ , the presentation (25) has the form  $b = \begin{pmatrix} 0 & 0 \\ 0 & b_{22} \end{pmatrix}$ .

The algebras  $e'_d \mathbb{I}_1 e'_d$  will appear in Theorem 5.7.(3) as a large part of the centralizer  $\text{Cen}_{\mathbb{I}_1}(a)$  for some elements  $a \in \mathbb{I}_1$ . The properties of the centralizer  $\text{Cen}_{\mathbb{I}_1}(a)$  of being a finitely generated algebra or a (left or right) Noetherian algebra largely depend on similar properties of the algebras  $e'_d \mathbb{I}_1 e'_d$ . Let us prove some results on the algebras  $e'_d \mathbb{I}_1 e'_d$  that will be used in the proof of Theorem 5.7. We can easily verify that

$$e'_d \partial^i e'_d = e'_d \partial^i, \quad e'_d \int^i e'_d = \int^i e'_d, \quad i \in \mathbb{N}. \quad (26)$$

The map  $\mathbb{I}_1 \rightarrow e'_d \mathbb{I}_1 e'_d$ ,  $a \mapsto e'_d a e'_d$ , is not an algebra homomorphism and neither an injective map as  $e'_d e_{00} e'_d = 0$ ; but its restriction to the subalgebra  $D_1$  yields the  $K$ -algebra epimorphism  $D_1 \rightarrow e'_d D_1 e'_d$ ,  $a \mapsto e'_d a e'_d$ , with kernel  $\bigoplus_{i=0}^d Ke_{ii}$ . Notice that  $e'_d D_1 e'_d = e'_d D_1 = D_1 e'_d$  is a commutative non-Noetherian algebra as  $\bigoplus_{i>d} Ke_{ii}$  is the direct sum of nonzero ideals  $Ke_{ii}$  of the algebra  $D_{1,d} := e'_d D_1 = K[e'_d H] \bigoplus \bigoplus_{i>d} Ke_{ii}$ .

**Lemma 5.6** *Let  $d \in \mathbb{N}$ . Then*

1. *The algebra  $e'_d \mathbb{I}_1 e'_d$  is a finitely generated algebra which is neither left nor right Noetherian algebra, and  $(e'_d \mathbb{I}_1 e'_d)^* = e'_d \mathbb{I}_1 e'_d$ . Moreover, the algebra  $e'_d \mathbb{I}_1 e'_d$  is generated by the elements  $e'_d \partial$ ,  $\int e'_d$ ,  $e'_d H$  and  $e_{d+1,d+1}$ ; and it contains infinite direct sums of nonzero left and right ideals.*
2. *The algebra  $e'_d \mathbb{I}_1 e'_d$  is a  $\mathbb{Z}$ -graded algebra which is a homogeneous subring of the  $\mathbb{Z}$ -graded algebra  $\mathbb{I}_1$ :*

$$e'_d \mathbb{I}_1 e'_d = \bigoplus_{i \geq 1} D_{1,d} \partial^i \bigoplus D_{1,d} \bigoplus \bigoplus_{i \geq 1} \int^i D_{1,d}$$

where  $D_{1,d} := e'_d D_1$ .

3. *Let  $\alpha \in K[H] \setminus K$ . Then  $\text{Cen}_{e'_d \mathbb{I}_1 e'_d}(e'_d \alpha) = e'_d \text{Cen}_{\mathbb{I}_1}(\alpha) e'_d = D_{1,d} \bigoplus C_{\alpha,d} \bigoplus C_{\alpha,d}^*$  where  $C_{\alpha,d} := \bigoplus_{i \geq 1, j > d} \{Ke_{i+j,j} \mid j+1 \text{ is a root of the polynomial } \tau^i(\alpha) - \alpha\}$ .*

*Proof.* 2. The element  $e'_d \in D_1$  is a homogeneous element of the  $\mathbb{Z}$ -graded algebra  $\mathbb{I}_1$  of graded degree 0. Therefore, the algebra  $e'_d \mathbb{I}_1 e'_d$  is a homogeneous subring of the algebra  $\mathbb{I}_1$ , and, by (4) and (26),

$$e'_d \mathbb{I}_1 e'_d = e'_d \left( \sum_{i \geq 1} D_{1,d} \partial^i + D_1 + \sum_{i \geq 1} \int^i D_{1,d} \right) e'_d = \sum_{i \geq 1} D_{1,d} \partial^i + D_{1,d} + \sum_{i \geq 1} \int^i D_{1,d},$$

and statement 2 follows.

1. By (26),  $(e'_d \partial)^i = e'_d \partial^i$  and  $(\int e'_d)^i = \int^i e'_d$  for all  $i \geq 1$ . For all  $i, j > d$ ,

$$e_{ij} = \int^{i-d-1} e_{d+1,d+1} \partial^{i-d-1} = \int^{i-d-1} e'_d e_{d+1,d+1} e'_d \partial^{i-d-1} = (\int e'_d)^{i-d-1} e_{d+1,d+1} (e'_d \partial)^{i-d-1}.$$

Therefore, by statement 2, the algebra  $e'_d \mathbb{I}_1 e'_d$  is generated by the elements  $e'_d \partial$ ,  $\int e'_d$ ,  $e'_d H$  and  $e_{d+1,d+1}$ . Since  $(e'_d)^* = e'_d$ , we have  $(e'_d \mathbb{I}_1 e'_d)^* = e'_d \mathbb{I}_1 e'_d$ . The sum  $\bigoplus_{j>d} E_{>d,j}$  where  $E_{>d,j} := \bigoplus_{i>d} K e_{ij}$  (resp.  $\bigoplus_{i>d} E_{i,>d}$  where  $E_{i,>d} := \bigoplus_{j>d} K e_{ij}$ ) is an infinite direct sum of nonzero left ideals  $E_{>d,j}$  (resp. right ideals  $E_{i,>d}$ ) of the algebra  $e'_d \mathbb{I}_1 e'_d$ . Therefore, the algebra  $e'_d \mathbb{I}_1 e'_d$  is neither left nor right Noetherian.

3. Since the elements  $e'_d$  and  $\alpha$  commute we have the inclusion  $C' := \text{Cen}_{e'_d \mathbb{I}_1 e'_d}(e'_d \alpha) \supseteq e'_d \text{Cen}_{\mathbb{I}_1}(\alpha) e'_d$ . By Proposition 5.1.(1),  $e'_d \text{Cen}_{\mathbb{I}_1}(\alpha) e'_d = D_{1,\alpha} \oplus C_{\alpha,d} \oplus C_{\alpha,d}^*$ . The element  $e'_d \alpha \in D_{1,d}$  is a homogeneous element of the  $\mathbb{Z}$ -graded algebra  $e'_d \mathbb{I}_1 e'_d$ . Since  $\pi(e'_d \alpha) = \alpha$  where  $\pi : \mathbb{I}_1 \rightarrow \mathbb{I}_1/F = B_1$ ,  $a \mapsto a + F$ , we see that  $\pi(\text{Cen}_{e'_d \mathbb{I}_1 e'_d}(e'_d \alpha)) \subseteq \text{Cen}_{B_1}(\alpha) = K[H]$  (see the proof of Proposition 5.1.(1)). Since  $e'_d K[H] \subseteq D_{1,d} \subseteq C'$  and  $\text{Cen}_{B_1}(\alpha) = K[H] = \pi(K[e'_d H])$ , we see that  $C' = D_{1,d} \oplus C_{\alpha,d} \oplus C_{\alpha,d}^*$ , and so  $C' = e'_d \text{Cen}_{\mathbb{I}_1}(e'_d \alpha) e'_d$ .  $\square$

Let  $K[t]$  be a polynomial algebra in a variable  $t$  over the field  $K$  and  $K(t)$  be its field of fractions. Let  $M$  be a  $K[t]$ -module. Then  $\text{tor}_{K[t]}(M) := \{m \in M \mid pm = 0 \text{ for some } p \in K[t] \setminus \{0\}\}$  is the  $K[t]$ -torsion submodule of  $M$ , it is the sum of all finite dimensional  $K[t]$ -submodules of  $M$ . The rank of the  $K[t]$ -submodule  $M$  is  $\dim_{K(t)}(K(t) \otimes_{K[t]} M)$ . For each non-scalar element of the algebra  $\mathbb{I}_1$ , the following theorem describes its centralizer.

**Theorem 5.7** 1. Let  $a \in \mathbb{I}_1 \setminus (K[H] + F)$ . Then

- (a) The centralizer  $\text{Cen}_{\mathbb{I}_1}(a)$  is a finitely generated  $K[a]$ -module of finite rank  $\rho \geq 1$ . In particular,  $\text{Cen}_{\mathbb{I}_1}(a)$  is a finitely generated, left and right Noetherian algebra.
- (b) Moreover, there is a  $K[a]$ -module isomorphism

$$\text{Cen}_{\mathbb{I}_1}(a) \simeq K[a]^\rho \bigoplus \text{Cen}_F(a),$$

and if  $-n = \pi_-(a) < 0$  (respectively,  $m = \pi_+(a) > 0$ ) then  $\rho$  divides  $n$  (respectively,  $m$ ).

- (c)  $[\text{Cen}_{\mathbb{I}_1}(a), \text{Cen}_{\mathbb{I}_1}(a)] \subseteq \text{Cen}_F(a)$ ,  $\dim_K(\text{Cen}_F(a)) < \infty$ , and  $\text{Cen}_F(a) = \text{tor}_{K[a]}(\text{Cen}_{\mathbb{I}_1}(a))$ .
2. Let  $a \in (K[H] + F) \setminus (K + F)$ , i.e.,  $a = \alpha + f$  for some polynomial  $\alpha \in K[H] \setminus K$  and  $f \in F$ ,  $d := \deg_F(a)$ . Then

- (a) The centralizer  $\text{Cen}_{\mathbb{I}_1}(a)$  is not a finitely generated  $K[a]$ -module but has finite rank  $\rho := \deg_H(\alpha)$  (as a  $K[a]$ -module).
- (b)  $\text{Cen}_{\mathbb{I}_1}(a)$  is not a finitely generated algebra and neither a left nor right Noetherian algebra. Moreover, it contains infinite direct sums of nonzero left and right ideals.
- (c) There is a  $K[a]$ -module isomorphism

$$\text{Cen}_{\mathbb{I}_1}(a) \simeq K[a]^\rho \bigoplus \text{Cen}_F(a).$$

- (d)  $\text{Cen}_{\mathbb{I}_1}(a) = e'_d K[H] \oplus \text{Cen}_F(a)$  and

$$\begin{aligned} \text{Cen}_F(a) = & \bigoplus_{j>d} K e_{jj} \oplus C_{\alpha,d} \oplus C_{\alpha,d}^* \oplus \text{Cen}_{F_{\leq d}}(a_{11}) \oplus \\ & \left( \bigoplus_{j>d} \ker_{\mathcal{H}_0}((a_{11} - \alpha(j+1)) \cdot) e_{0j} \right) \oplus \bigoplus_{i>d} e_{i0} \ker_{\mathcal{V}_0}(\cdot (a_{11} - \alpha(i+1))) \end{aligned}$$

is an infinite direct sum of  $K[a]$ -modules where  $a_{11} := \sum_{i=0}^d \alpha(i+1)e_{ii} + f$ ,  $F_{\leq d} := \bigoplus_{i,j=0}^d Ke_{ij}$ ,  $\mathcal{H}_0 := \bigoplus_{i=0}^d Ke_{i0}$ ,  $\mathcal{V}_0 := \bigoplus_{j=0}^d Ke_{0j}$  and  $C_{\alpha,d} := \bigoplus_{i \geq 1, j > d} \{Ke_{i+j,j} \mid j+1 \text{ is a root of the polynomial } \tau^i(\alpha) - \alpha\}$ .

(e)  $[\text{Cen}_{\mathbb{I}_1}(a), \text{Cen}_{\mathbb{I}_1}(a)] \subseteq \text{Cen}_F(a)$ ,  $\dim_K(\text{Cen}_F(a)) = \infty$ ,  $\text{Cen}_F(a) = \text{tor}_{K[a]}(\text{Cen}_{\mathbb{I}_1}(a))$ .

3. Let  $a \in (K+F) \setminus K$ , i.e.,  $a = \lambda + f$  for some elements  $\lambda \in K$  and  $f = \sum_{i,j=0}^d \lambda_{ij}e_{ij} \in F \setminus \{0\}$  where  $d := \deg_F(f)$ ; let  $e_d = e_{00} + \dots + e_{dd}$  and  $e'_d = 1 - e_d$ . Then

- (a) The centralizer  $\text{Cen}_{\mathbb{I}_1}(a)$  is a finitely generated algebra which is neither a left nor right Noetherian algebra. Moreover, it contains infinite direct sums of nonzero left and right ideals. In particular,  $\text{Cen}_{\mathbb{I}_1}(a)$  is not a finitely generated  $K[a]$ -module.
- (b)  $\text{Cen}_{\mathbb{I}_1}(a) = \text{Cen}_{F_{\leq d}}(f) \oplus \bigoplus_{j>d} \mathcal{K}e_{0j} \oplus \bigoplus_{i>d} e_{i0}\mathcal{K}' \oplus e'_d\mathbb{I}_1e'_d$  where  $\mathcal{K} := \ker_{\bigoplus_{i=0}^d Ke_{i0}}(f \cdot)$  and  $\mathcal{K}' := \ker_{\bigoplus_{j=0}^d Ke_{0j}}(\cdot f)$ .
- (c)  $[\text{Cen}_{\mathbb{I}_1}(a), \text{Cen}_{\mathbb{I}_1}(a)] \not\subseteq \text{Cen}_F(a)$ ,  $\dim_K(\text{Cen}_F(a)) = \infty$ ,  $\text{tor}_{K[a]}(\text{Cen}_{\mathbb{I}_1}(a)) = \text{Cen}_{\mathbb{I}_1}(a) \neq \text{Cen}_F(a)$ .

*Proof.* 1. Since  $a \notin K[H] + F$  then either  $-n := \pi_-(a) < 0$  or  $m := \pi_+(a) > 0$  (or both). In view of  $(\pm)$ -symmetry, let us assume that  $-n := \pi_-(a) < 0$ , i.e.,  $a = \alpha\partial^n + \dots$  where  $\alpha\partial^n$  is the least term of the element  $a$  modulo  $F$  and  $\alpha \in K[H] \setminus \{0\}$ . Let  $C := \text{Cen}_{\mathbb{I}_1}(a)$ . We claim that

$$C \cap \mathbb{I}_{1,+} \subseteq F \quad (27)$$

where  $\mathbb{I}_{1,+} := \bigoplus_{i \geq 1} K[H] \int^i \oplus F$ . If  $\alpha \in K[H] \setminus K$  this follows from (23) and Corollary 5.5.(2). If  $\alpha \in K \setminus \{0\}$  then we may assume that  $\alpha = 1$ , by multiplying the element  $a$  by  $\alpha^{-1}$ . Suppose that the inclusion (27) fails, and so there is an element  $c \in C \cap \mathbb{I}_{1,+}$  with  $l := \pi_-(c) \geq 1$ . Let  $\beta \int^l$  be the least term of the element  $c$  modulo  $F$  where  $\beta \in K[H] \setminus \{0\}$ . Since  $\text{Cen}_{B_1}(\pi(a) = \partial^n) = K[\partial, \partial^{-1}]$ , we must have  $\beta \in K^*$ . Without loss of generality we may assume that  $\beta = 1$  (by dividing  $c$  by  $\beta$ ). According to (7), the elements  $a$  and  $c$  can be uniquely written as the sums  $a = \partial^n + a' + f$  and  $c = \int^l + c' + g$  where  $f, g \in F$ ,  $a'$  is the sum  $\sum b_i v_i$  in the decomposition (7) without the least term  $\partial^n$ , and similarly  $c'$  is defined. Replacing  $a$  and  $c$  by  $a'$  and  $c'$  respectively we may assume that  $n = l$  in the presentations above, i.e.,  $a = \partial^n + a' + f$  and  $c = \int^n + c' + g$ . Notice that  $\xi([\partial^n, c']) = \xi([a', \int^n]) = \xi([a', c']) = 0$  as the elements in the brackets are sums of homogeneous elements of the  $\mathbb{Z}$ -graded algebra  $\mathbb{I}_1$  of positive graded degrees. Clearly,  $\text{tr} \circ \xi|_F = \text{tr}$ . By (20),  $\text{tr} \circ \xi([F, \mathbb{I}_1]) = \text{tr}([F, \mathbb{I}_1]) = 0$ . Now, applying the map  $\text{tr} \circ \xi$  to the equality

$$0 = [a, c] = [\partial^n, \int^n] + [a', \int^n] + [f, \int^n] + [\partial^n, c'] + [a', c'] + [f, c'] + [a, g]$$

we get a contradiction:

$$0 = \text{tr} \circ \xi([\partial^n, \int^n]) = \text{tr} \circ \xi(e_{00} + e_{11} + \dots + e_{n-1, n-1}) = n.$$

Therefore, (27) holds. The set  $C \setminus C_F$  is a multiplicative monoid where  $C_F := \text{Cen}_F(a)$ . We denote by  $\kappa$  the composition of the two monoid homomorphisms  $\pi_- : C \setminus C_F \rightarrow -\mathbb{N}$  (see (27)) and  $\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$ , and so  $\kappa$  is a monoid homomorphism ( $\kappa(uv) = \kappa(u) + \kappa(v)$  and  $\kappa(1) = 0$ ). Therefore, its image  $G := \text{im}(\kappa)$  is a cyclic subgroup of order, say  $\rho$ , and  $\rho|n$ . Let  $G = \{m_1 = 0, m_2, \dots, m_\rho\}$ . Then for each element  $m_i$  we choose an element  $g_i \in C$  so that  $\kappa(g_i) = m_i$  and the number  $\pi_-(g_i) \in -\mathbb{N}$  is the largest possible integer. We may choose  $g_1 = 1$ , by (27). Let us show that the  $K[a]$ -submodule  $M := \sum_{i=1}^\rho K[a]g_i$  of  $C$  is free, i.e.,  $M = \bigoplus_{i=1}^\rho K[a]g_i$ , and  $C = M \oplus C_F$ .

Suppose that  $\varphi_1 g_1 + \dots + \varphi_\rho g_\rho \in F$  for some elements  $\varphi_i \in K[a]$ , not all of which are equal to zero, we seek a contradiction. Then there exist nonzero terms  $\varphi_i g_i$  and  $\varphi_j g_j$  such that  $\pi_-(\varphi_i g_i) = \pi_-(\varphi_j g_j)$ , and so  $\kappa(g_i) = \kappa(g_j)$ . This contradicts to the choice of the elements  $g_1, \dots, g_\rho$ . Therefore,  $M = \bigoplus_{i=1}^\rho K[a]g_i$  and  $M \cap C_F = 0$ . To finish the proof the claim it remains

to show that  $C = M + C_F$ . Choose an arbitrary nonzero element  $g \in C$ . If  $\pi_-(g) = 0$  then, by Corollary 5.5.(1) and (27),  $g \in K + C_F = Kg_1 + C_F$ . If  $\pi_-(g) = k < 0$  then there exists  $g_i$  such that  $\kappa(g) = \kappa(g_i)$ , and so  $k = \pi_-(a^s g_i)$  for some natural number  $s \in \mathbb{N}$  (by the choice of the elements  $g_1, \dots, g_\rho$ ). By Corollary 5.5.(1), there exists  $\lambda \in K$  such that  $\pi_-(g - \lambda a^s g_i) > k$ . Using induction on  $|k|$  or repeating the same argument several times we see that  $C = M + C_F$ . Therefore,  $C = M \oplus C_F$  where  $C_F$  is a finite dimensional ideal of the algebra  $C$  (Proposition 5.1.(2)). This implies that  $\rho$  is the rank of the  $K[a]$ -module  $C$  and  $\text{Cen}_F(a) = \text{tor}_{K[a]}(C)$ . In particular,  $C$  is a finitely generated  $K[a]$ -module, and so  $C$  is a finitely generated left and right Noetherian algebra.

Let us show that  $uv - vu \in C_F$  for all elements  $u, v \in C$ . It suffices to show that  $uv - vu \in F$  as  $uv - vu \in C$  and  $C_F = C \cap F$ . Choose an element  $g \in C$  such that  $\kappa(g)$  is a generator for the group  $G$ . Denote by  $E$  the subalgebra (necessarily commutative) of  $C$  generated by the elements  $a$  and  $g$ . Notice that  $C_F$  is an ideal of the algebra  $C$ . By the choice of  $g$ , the  $K[a]$ -module  $C/(E + C_F)$  is *finite dimensional*. Therefore, there exist nonzero elements  $P, Q \in K[a]$  such that  $Pu, Qv \in E + C_F$ . Then

$$PQuv \equiv (Pu)(Qv) \equiv (Qv)(Pu) \equiv PQvu \pmod{C_F},$$

i.e.,  $PQ(uv - vu) \in C_F$ . Since  $PQ$  is a nonzero element of the algebra  $K[a]$  (which is isomorphic to a polynomial algebra over  $K$  in a single variable, since  $a \notin F$ ) and  $F \cap K[a] = 0$ , we see that  $PQ \notin F$ , then  $uv - vu \in C_F$  since the algebra  $B_1$  is a domain.

2. Let  $C := \text{Cen}_{\mathbb{I}_1}(a)$  and  $C_F := \text{Cen}_F(a)$ . Let  $b \in \mathbb{I}_1$ . By (25),  $b = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$  and  $a = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}$  where  $a_{11} := \sum_{i=0}^d \alpha(i+1)e_{ii} + f$  and  $\alpha(i+1)$  is the value of the polynomial  $\alpha(H)$  at  $H = i+1$ ,  $a_{22} = e'_d a e'_d = e'_d \alpha = \alpha e'_d \in e'_d \mathbb{I}_1 e'_d$ . Then  $b \in C$  iff  $\begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}$  iff  $b_{11} \in \text{Cen}_{F_{\leq d}}(a_{11})$ ,  $b_{22} \in \text{Cen}_{e'_d \mathbb{I}_1 e'_d}(e'_d \alpha)$ ,  $a_{11} b_{12} = b_{12} a_{22}$  and  $a_{22} b_{21} = b_{21} a_{11}$ . Notice that  $\begin{pmatrix} b_{11} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b_{12} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ b_{21} & 0 \end{pmatrix} \in C_F$ . Then, by Lemma 5.6.(3),

$$C = e'_d K[H] \bigoplus C_F \quad (28)$$

and  $e'_d K[H]$  is a free  $K[a]$ -module (i.e.,  $K[\alpha]$ -module since  $a e'_d = \alpha e'_d$ ) of rank  $\rho = \deg_H(\alpha)$ . As a  $K[a]$ -bimodule (in particular, as a  $K[\text{ad}(a)]$ -module where  $\text{ad}(a) := [a, \cdot]$  is the inner derivation of the algebra  $\mathbb{I}_1$ ) the ideal  $F$  is the direct sum of four sub-bimodules:

$$F_{\leq d}, \quad F_{> d} := \bigoplus_{i,j>d} K e_{ij}, \quad \mathcal{H} := \bigoplus_{i=0}^d \bigoplus_{j>d} K e_{ij}, \quad \mathcal{V} := \bigoplus_{i>d} \bigoplus_{j=0}^d K e_{ij}.$$

Then

$$\text{Cen}_F(a) = \ker_{F_{\leq d}}(\text{ad}(a)) \bigoplus \ker_{F_{> d}}(\text{ad}(a)) \bigoplus \ker_{\mathcal{H}}(\text{ad}(a)) \bigoplus \ker_{\mathcal{V}}(\text{ad}(a)).$$

$\ker_{F_{\leq d}}(\text{ad}(a))$  is the centralizer of the  $(d+1) \times (d+1)$  matrix  $a_{11}$  in the algebra  $F_{\leq d}$  of  $(d+1) \times (d+1)$  matrices.  $\ker_{F_{> d}}(\text{ad}(a)) = \ker_{F_{> d}}(\text{ad}(\alpha)) = \bigoplus_{j>d} K e_{jj} \bigoplus C_{\alpha,d} \bigoplus C_{\alpha,d}^*$ , by Proposition 5.1.(1). The  $K[a]$ -bimodule  $\mathcal{H} = \bigoplus_{j>d} \mathcal{H}_j$  is the direct sum of finite dimensional  $K[a]$ -bimodules  $\mathcal{H}_j := \bigoplus_{i=0}^d K e_{ij}$ , and the action of the map  $\text{ad}(a)$  on  $\mathcal{H}_j$  is equal to  $(a_{11} - \alpha(j+1))\mathcal{H}_j$ . Therefore,

$$\begin{aligned} \ker_{\mathcal{H}_j}(\text{ad}(a)) &= \ker_{\mathcal{H}_j}((a_{11} - \alpha(j+1))\cdot) = \ker_{\mathcal{H}_0}((a_{11} - \alpha(j+1))\cdot) e_{0j}, \\ \ker_{\mathcal{H}}(\text{ad}(a)) &= \bigoplus_{j>d} \ker_{\mathcal{H}_0}((a_{11} - \alpha(j+1))\cdot) e_{0j}. \end{aligned}$$

Similarly, the  $K[a]$ -bimodule  $\mathcal{V} = \bigoplus_{i>d} \mathcal{V}_i$  is the direct sum of finite dimensional  $K[a]$ -bimodules  $\mathcal{V}_i = \bigoplus_{j=0}^d Ke_{ij}$  and the action of the map  $\text{ad}(a)$  on  $\mathcal{V}_i$  is equal to  $\cdot(a_{11} - \alpha(i+1))$ . Therefore,

$$\begin{aligned} \ker_{\mathcal{V}_i}(\text{ad}(a)) &= \ker_{\mathcal{V}_i}(\cdot(a_{11} - \alpha(i+1))) = e_{i0}\ker_{\mathcal{V}_0}(\cdot(a_{11} - \alpha(i+1))), \\ \ker_{\mathcal{V}}(\text{ad}(a)) &= \bigoplus_{i>d} e_{i0}\ker_{\mathcal{V}_0}(\cdot(a_{11} - \alpha(i+1))). \end{aligned}$$

It is obvious that  $\text{Cen}_F(a)$  is a  $K[a]$ -torsion, infinite dimensional, not finitely generated  $K[a]$ -module. Therefore, by (28),  $C$  is not a finitely generated  $K[a]$ -module,  $\text{tor}_{K[a]}(C) = C_F$ , and the  $K[a]$ -module  $C$  has rank  $\rho = \deg_H(\alpha)$ . By (28),  $[C, C] \subseteq C_F$ . Recall that the left  $\mathbb{I}_1$ -module  $F$  is the direct sum  $\bigoplus_{j \in \mathbb{N}} E_{\mathbb{N},j}$  of nonzero left ideals  $E_{\mathbb{N},j} = \bigoplus_{i \in \mathbb{N}} Ke_{ij}$ . Similarly, the right  $\mathbb{I}_1$ -module  $F$  is the direct sum  $\bigoplus_{i \in \mathbb{N}} E_{i,\mathbb{N}}$  of nonzero right ideals  $E_{i,\mathbb{N}} = \bigoplus_{j \in \mathbb{N}} Ke_{ij}$ . Since

$$e_{jj} \in E_{\mathbb{N},j} \cap \text{Cen}_{\mathbb{I}_1}(a) \text{ for all } j > d; \quad e_{ii} \in E_{i,\mathbb{N}} \cap \text{Cen}_{\mathbb{I}_1}(a) \text{ for all } i > d, \quad (29)$$

the sums  $\bigoplus_{j>d} (E_{\mathbb{N},j} \cap \text{Cen}_{\mathbb{I}_1}(a))$  and  $\bigoplus_{i>d} (E_{i,\mathbb{N}} \cap \text{Cen}_{\mathbb{I}_1}(a))$  are infinite direct sums of nonzero left and right ideals of the algebra  $C$  respectively. Therefore, the algebra  $C$  is neither left nor right Noetherian. To finish the proof of statement 2 it remains to show that the algebra  $C$  is not finitely generated. Suppose that  $S$  is a finite set of algebra generators for the algebra  $C$ , we seek a contradiction. We may assume that  $e'_d H \in S$ . Then, by (28), we may assume that  $S = \{e'_d H\} \cup S_F$  where  $S_F$  is a finite subset of  $C_F$ . Then we can fix a natural number  $n$  such that  $S_F \subseteq F_{\leq n}$ . Then  $C = K\langle S \rangle \subseteq K\langle F_{\leq n}, e'_d H \rangle = K[e'_d H] \oplus F_{\leq n}$ , and so  $C_F \subseteq F_{\leq n}$ , a contradiction, since  $e_{n+1,n+1} \in C_F \setminus F_{\leq n}$ .

3. Let  $C := \text{Cen}_{\mathbb{I}_1}(a)$  and  $C_F := \text{Cen}_F(a)$ . Since  $C = \text{Cen}_{\mathbb{I}_1}(f)$ , we may assume that  $a = f \in F \setminus K$ . Let  $R$  be the RHS in the equality in statement 3(b). Let  $b \in \mathbb{I}_1$ . By (25),  $b = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$  and  $f = \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}$ . Then  $b \in C$  iff  $\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}$  iff  $fb_{11} = b_{11}f$ ,  $fb_{12} = 0$  and  $b_{12}f = 0$  iff  $b \in R$ . It follows from the equality  $C = R$  and Lemma 5.6 that the centralizer  $C$  is a finitely generated algebra which is generated by the finite dimensional subspaces  $\text{Cen}_{F_{\leq d}}(f)$ ,  $\mathcal{K}e_{0,d+1}$ ,  $e_{d+1,0}\mathcal{K}'$  and the elements  $e'_d$ ,  $e'_d\partial$ ,  $\int e'_d$  and  $e_{d+1,d+1}$  (since, for all  $i > d$ ,  $\mathcal{K}e_{0i} = \mathcal{K}e_{0,d+1}(e'_d\partial)^{i-d-1}$  and  $e_{i0}\mathcal{K}' = (\int e'_d)^{i-d-1}e_{d+1,0}\mathcal{K}'$ ). Clearly,  $[C, C] \not\subseteq C_F$  since

$$[e'_d H e'_d, e'_d \partial e'_d] \equiv [H, \partial] \equiv -\partial \pmod{F},$$

and  $\dim_K(C) \geq \dim_K(e'_d D_1) = \infty$ . By (29), the sums  $\bigoplus_{j>d} (E_{\mathbb{N},j} \cap \text{Cen}_{\mathbb{I}_1}(a))$  and  $\bigoplus_{i>d} (E_{i,\mathbb{N}} \cap \text{Cen}_{\mathbb{I}_1}(a))$  are infinite direct sums of nonzero left and right ideals of the algebra  $C$  respectively. Therefore, the algebra  $C$  is neither left nor right Noetherian. Clearly,  $\text{tor}_{K[a]}(C) = C \neq C_F$ .  $\square$

**Corollary 5.8** *Let  $a \in \mathbb{I}_1 \setminus K$ . Then the following statements are equivalent.*

1.  $a \notin K[H] + F$ .
2.  $\text{Cen}_{\mathbb{I}_1}(a)$  is a finitely generated  $K[a]$ -module.
3.  $\text{Cen}_{\mathbb{I}_1}(a)$  is a left Noetherian algebra.
4.  $\text{Cen}_{\mathbb{I}_1}(a)$  is a right Noetherian algebra.
5.  $\text{Cen}_{\mathbb{I}_1}(a)$  is a finitely generated and Noetherian algebra.

**Corollary 5.9** *Let  $a \in \mathbb{I}_1 \setminus K$ . Then*

1.  $\text{Cen}_{\mathbb{I}_1}(a)$  is a finitely generated algebra iff  $a \notin (K[H] + F) \setminus (K + F)$ .
2.  $\text{Cen}_{\mathbb{I}_1}(a)$  is a finitely generated not Noetherian/not left Noetherian/not right Noetherian algebra iff  $a \in (K + F) \setminus K$ .
3. The algebra  $\text{Cen}_{\mathbb{I}_1}(a)$  is not finitely generated, not Noetherian/not left Noetherian/not right Noetherian iff  $a \in (K[H] + F) \setminus (K + F)$ .



## 6 The simple $\mathbb{I}_1$ -module $K[x]$ and a classification of elements of the algebra $\mathbb{I}_1$

In this section, a formula for the index  $\text{ind}_{K[x]}(a)$  of elements  $a \in \mathbb{I}_1 \setminus F$  is found (Proposition 6.1). Recall that  $K[x]$  is the unique (up to isomorphism) *faithful* simple  $\mathbb{I}_1$ -module and  $\mathbb{I}_1 \subseteq \text{End}_K(K[x])$ . Classifications of elements  $a \in \mathbb{I}_1$  are given such that the map  $a_{K[x]} : K[x] \rightarrow K[x]$ ,  $p \mapsto ap$ , is a bijection (Theorem 6.2), a surjection (Theorem 6.3) or an injection (Theorem 6.6). In case when the map  $a_{K[x]}$  is a bijection, an explicit inversion formula is found (Theorem 6.2.(4)). As a result we have a formula for the unique polynomial solution  $q = a^{-1} * p$  of the polynomial integro-differential operator equation  $a * p = q$  where  $p, q \in K[x]$  and  $a \in \mathbb{I}_1^0 := \mathbb{I}_1 \cap \text{Aut}_K(K[x])$ . The monoid  $\mathbb{I}_1^0$  is much more massive set than the group  $\mathbb{I}_1^*$  of units of the algebra  $\mathbb{I}_1$  (Theorem 6.2.(1,2), Theorem 8.3.(6)). In case when the map  $a_{K[x]}$  is surjective or injective we found its kernel and cokernel respectively (Theorem 6.3 and Theorem 6.6).

Each nonzero element  $u$  of the skew Laurent polynomial algebra  $B_1 = K[H][\partial, \partial^{-1}; \tau]$  (where  $\tau(H) = H + 1$ ) is the unique sum  $u = \alpha_s(\partial^{-1})^s + \alpha_{s+1}(\partial^{-1})^{s+1} + \dots + \alpha_d(\partial^{-1})^d$  where all  $\alpha_i \in K[H]$ ,  $\alpha_d \neq 0$ , and  $\alpha_d(\partial^{-1})^d$  is the *leading term* of the element  $u$ . The integer  $\deg_{\partial^{-1}}(u) = d$  is called the *degree* of the element  $u$  in the noncommutative variable  $\partial^{-1}$ ,  $\deg_{\partial^{-1}}(0) := -\infty$ . For all elements  $u, v \in B_1$ ,  $\deg_{\partial^{-1}}(uv) = \deg_{\partial^{-1}}(u) + \deg_{\partial^{-1}}(v)$ ,  $\deg_{\partial^{-1}}(u+v) \leq \max\{\deg_{\partial^{-1}}(u), \deg_{\partial^{-1}}(v)\}$ . Therefore, the minus degree function

$$-\deg_{\partial^{-1}} : \text{Frac}(B_1) \rightarrow \mathbb{Z} \cup \{\infty\}, \quad s^{-1}u \mapsto \deg_{\partial^{-1}}(s) - \deg_{\partial^{-1}}(u),$$

is a discrete valuation on the skew field of fractions  $\text{Frac}(B_1)$  of the algebra  $B_1$  such that  $\deg_{\partial^{-1}}(\alpha) = 0$  for all elements  $0 \neq \alpha \in K(H)$ .

**Proposition 6.1** *Let  $a \in \mathbb{I}_1 \setminus F$ .*

1.  $\text{ind}_{K[x]}(a) = -\deg_{\partial^{-1}}(a + F)$  where  $a + F \in \mathbb{I}_1/F = B_1$ .
2. Let  $i = \text{ind}_{K[x]}(a)$ . Then
  - (a)  $i \geq 0$  iff  $a = \partial^i a'$  for some element  $a' \in \mathbb{I}_1 \setminus F$  with  $\text{ind}_{K[x]}(a') = 0$ .
  - (b)  $i \leq 0$  iff  $a = a' \int^{[i]}$  for some element  $a' \in \mathbb{I}_1 \setminus F$  with  $\text{ind}_{K[x]}(a') = 0$ .
3. Let  $b \in \mathbb{I}_1 \setminus F$  such that  $a + b \in \mathbb{I}_1 \setminus F$ . Then  $\text{ind}_{K[x]}(a + b) \geq \min\{\text{ind}_{K[x]}(a), \text{ind}_{K[x]}(b)\}$ . Therefore, the index map  $\text{ind}_{K[x]} : \text{Frac}(A_1) \rightarrow \mathbb{Z} \cup \{\infty\}$  is a discrete valuation.
4.  $\text{ind}_{K[x]}(\sigma(a)) = \text{ind}_{K[x]}(a)$  for all automorphisms  $\sigma \in \text{Aut}_{K\text{-alg}}(\mathbb{I}_1)$ .

*Remark.* In statement 2(a),  $a = \partial^i a'$  ( $i \geq 1$ ) does not imply  $a = a'' \partial^i$  for some element (necessarily)  $a'' \in \mathbb{I}_1 \setminus F$  with  $\text{ind}_{K[x]}(a'') = 0$ ; eg,  $a = \partial^i(1 + e_{i0})$  since  $e_{00} \notin \text{im}(\cdot \partial^i)$ . Similarly, in statement 2(b),  $a = a' \int^{[i]}$  ( $i \leq -1$ ) does not imply  $a = \int^{[i]} a''$  for some element (necessarily)  $a'' \in \mathbb{I}_1 \setminus F$  with  $\text{ind}_{K[x]}(a'') = 0$ ; eg,  $a = (1 + e_{0i}) \int^{[i]}$  since  $e_{00} \notin \text{im}(\int^{[i]} \cdot)$ .

*Proof.* 1. Let  $a$  be as in (5) and  $d = -\deg_{\partial^{-1}}(a + F)$ . By Lemma 3.5, without loss of generality we may assume that all  $\lambda_{ij} = 0$ . Notice that the maps  $\int^i \cdot, \cdot \partial^i : \mathbb{I}_1 \rightarrow \mathbb{I}_1$  are injections for all natural numbers  $i \geq 1$  (since  $\partial^i \int^i = 1$ ) such that  $\int^i(\mathbb{I}_1 \setminus F) \subseteq \mathbb{I}_1 \setminus F$  and  $(\mathbb{I}_1 \setminus F) \partial^i \subseteq \mathbb{I}_1 \setminus F$ . If  $d \geq 0$  then  $a = \partial^d \int^d a$  and  $\text{ind}_{K[x]}(a) = \text{ind}_{K[x]}(\partial^d) + \text{ind}_{K[x]}(\int^d a) = d + \text{ind}_{K[x]}(\int^d a)$ . If  $d \leq 0$  then  $a = a \partial^{-d} \int^{-d}$  and  $\text{ind}_{K[x]}(a) = \text{ind}_{K[x]}(\int^{-d}) + \text{ind}_{K[x]}(a \partial^{-d}) = d + \text{ind}_{K[x]}(a \partial^{-d})$ . Therefore, to finish the proof of statement 1 it suffices to show that if  $d = 0$  then  $\text{ind}_{K[x]}(a) = 0$ . Since  $d = 0$ ,  $a = \sum_{i \geq 1} a_{-i} \partial^i + a_0$  with  $a_0 \neq 0$ . The element  $a$  respects the finite dimensional filtration  $\{K[x]_{\leq i}\}_{i \in \mathbb{N}}$  of the  $\mathbb{I}_1$ -module  $K[x]$ , that is,  $aK[x]_{\leq i} \subseteq K[x]_{\leq i}$  for all  $i \in \mathbb{N}$ . Notice that  $a_0(H) * x^i = a_0(i+1)x^i$  for all  $i \in \mathbb{N}$ . Fix a natural number, say  $N$ , such that  $a_0(i+1) \neq 0$  for all  $i > N$ . Then the linear map  $a \cdot : V := K[x]/K[x]_{\leq N} \rightarrow V$  is a bijection. It follows from

the short exact sequence of  $K[a]$ -modules  $0 \rightarrow K[x]_{\leq N} \rightarrow K[x] \rightarrow V \rightarrow 0$  that  $\text{ind}_{K[x]_{\leq N}}(a) = \text{ind}_{K[x]_{\leq N}}(a) + \text{ind}_V(a) = 0 + 0 = 0$ .

2. Statement 2(a) (resp. 2(b)) follows from statement 1, (5), and the fact that  $\partial^i F = F$  (resp.  $F \int^{[i]} = F$  and  $\partial^{[i]} \int^{[i]} = 1$ ), see the proof of statement 1 where statement 2 was, in fact, proved.

3. Statement 3 follows from statement 1.

4. The simple  $\mathbb{I}_1$ -module  $K[x]$  is the only (up to isomorphism) *faithful* simple  $\mathbb{I}_1$ -module (Theorem 2.1). Therefore,  ${}^\sigma K[x] \simeq K[x]$  for all automorphisms  $\sigma$  of the algebra  $\mathbb{I}_1$ . Then  $\text{ind}_{K[x]}(\sigma(a)) = \text{ind}_{{}^\sigma K[x]}(a) = \text{ind}_{K[x]}(a)$ , by (11).  $\square$

**Classification of elements  $a \in \mathbb{I}_1$  such that  $a_{K[x]}$  is a bijection.** Let  $\text{Aut}_K(K[x])$  be the group of all invertible  $K$ -linear maps in the vector space  $K[x]$ , i.e., it is the group of units of the algebra  $\text{End}_K(K[x])$ . The next theorem describes the intersection  $\mathbb{I}_1^0 := \mathbb{I}_1 \cap \text{Aut}_K(K[x])$  and gives an inversion formula in  $\text{Aut}_K(K[x])$  for each element in the intersection. The set  $\mathbb{I}_1^0$  is a multiplicative monoid the elements of which are (left and right) regular elements of the algebra  $\mathbb{I}_1$  (i.e., non-zero-divisors). We will see that the multiplicatively closed set  $\mathbb{I}_1^0$  is the *largest* (with respect to inclusion) right Ore set that consists of regular elements (Theorem 9.7.(2)). It is obvious that if the map  $a_{K[x]}$  is a surjection (where  $a \in \mathbb{I}_1$ ) then  $a \notin F$ . Recall that  $\mathbb{I}_1^* = K^*(1 + F)^*$ , [12], where  $\mathbb{I}_1^*$  is the group of units of the algebra  $\mathbb{I}_1$ .

**Theorem 6.2** *Let  $a \in \mathbb{I}_1 \setminus F$  and  $d := \deg_F(a)$ . Then*

1.  $a_{K[x]} \in \text{Aut}_K(K[x])$  iff  $a = \sum_{i \geq 1} a_{-i} \partial^i + a_0 + \sum_{i,j \in \mathbb{N}} \lambda_{ij} e_{ij}$  (see (5)),  $b := a|_{K[x]_{\leq d}} \in \text{GL}(K[x]_{\leq d})$  and  $s+1$  is not a root of the nonzero polynomial  $a_0 \in K[H]$  for all natural numbers  $s > d$ .
2.  $\mathbb{I}_1^0 := \mathbb{I}_1 \cap \text{Aut}_K(K[x]) \supsetneq \mathbb{I}_1^* = K^*(1 + F)^*$ .
3. If  $\text{ind}_{K[x]}(a) = 0$  then the following statements are equivalent:
  - (a)  $a_{K[x]}$  is an injection,
  - (b)  $a_{K[x]}$  is a surjection,
  - (c)  $a_{K[x]}$  is a bijection.
4. (Inversion Formula) Suppose that  $a_{K[x]} \in \text{Aut}_K(K[x])$ ,  $a = a_- + a_0 + \sum \lambda_{ij} e_{ij}$  where  $a_- = \sum_{i \geq 1} a_{-i} \partial^i$ . Then according to the decomposition  $K[x] = K[x]_{\leq d} \oplus K[x]_{> d}$  where  $K[x]_{> d} := \bigoplus_{i > d} Kx^i$  the map  $a_{K[x]}$  is the matrix  $\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$  where  $a_{11} := a|_{K[x]_{\leq d}}$ ,  $a_{22} := a|_{K[x]/K[x]_{\leq d}} = (a_- + a_0)|_{K[x]/K[x]_{\leq d}}$ , the matrix of the linear map  $a_{12} : K[x]/K[x]_{\leq d} \rightarrow K[x]_{\leq d}$  has only finite number of nonzero entries with respect to the monomial bases of the vector spaces, and

$$\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}^{-1} = \begin{pmatrix} a_{11}^{-1} & -a_{11}^{-1} a_{12} a_{22}^{-1} \\ 0 & a_{22}^{-1} \end{pmatrix} \quad (30)$$

where  $a_{22}^{-1} = \sum_{i > 0} (-1)^i (a_0^{-1} a_-)^i a_0^{-1}$  (notice that  $a_0^{-1} a_-$  is a locally nilpotent map on the vector space  $K[x]/K[x]_{\leq d}$ ).

*Proof.* 1. Suppose that  $a_{K[x]} \in \text{Aut}_K(K[x])$ . Then  $\text{ind}_{K[x]}(a) = 0$ . By Proposition 6.1.(1),  $a = \sum_{i \geq 1} a_{-i} \partial^i + a_0 + \sum \lambda_{ij} e_{ij}$  with  $a_0 \neq 0$ . Since  $K[x]_{\leq d}$  is an  $a$ -invariant finite dimensional subspace, we must have  $b \in \text{GL}(K[x]_{\leq d})$ . The vector space  $K[x] = \bigcup_{s \geq d} K[x]_{\leq s}$  is the union of  $a$ -invariant finite dimensional subspace  $K[x]_{\leq s}$  and the element  $a$  acts on the one-dimensional factor space  $K[x]_{\leq s}/K[x]_{\leq s-1}$  by scalar multiplication  $a_0(s+1)$  (for all  $s > d$ ) since  $a_0(H) * x^s = a_0(s+1)x^s$ . Now, statement 1 is obvious.

2. Statement 2 is obvious (see statement 1).

3. For an arbitrary element  $a \in \mathbb{I}_1 \setminus F$  with  $\text{ind}_{K[x]}(a) = 0$ , we have seen in the proof of statement 1 that each term of the finite dimensional filtration  $\{K[x]_{\leq s}\}_{s \geq d}$  of the  $\mathbb{I}_1$ -module  $K[x]$  is an  $a$ -invariant subspace. For a linear endomorphism acting in a finite dimensional vector

space the conditions of being an injection, a surjection, or a bijection are equivalent. Therefore, statement 3 follows.

4. Clearly, (30) holds and  $a_{22}^{-1} = (a_0 + a_-)^{-1} = (a_0(1 + a_0^{-1}a_-))^{-1} = (1 + a_0^{-1}a_-)^{-1}a_0^{-1} = \sum_{i \geq 0} (-1)^i (a_0^{-1}a_-)^i a_0^{-1}$ .  $\square$

**Classification of the elements  $a \in \mathbb{I}_1$  such that  $a_{K[x]}$  is a surjection.**

**Theorem 6.3** *Let  $a \in \mathbb{I}_1$ . Then the map  $a_{K[x]}$  is surjective iff  $a = \sum_{i \geq n} a_{-i} \partial^i + a_F = a' \partial^n + a_F$  where  $n \geq 0$ ,  $a_{-n} \neq 0$ , all  $a_{-i} \in K[H]$ ,  $a_F \in F$ ,  $a' := \sum_{i \geq n} a_{-i} \partial^{i-n}$ , and*

1. *if  $a_F = 0$  then none of the natural numbers  $j \geq 1$  is a root of the polynomial  $a_{-n} \in K[H]$ ; in this case,  $a'_{K[x]}$  is a bijection and  $\ker(a_{K[x]}) = \ker(\partial_{K[x]}^n) = \bigoplus_{i=0}^{n-1} Kx^i$ ; and*
2. *if  $a_F \neq 0$  then none of the natural numbers  $j \geq d+2$  (where  $d = \deg_F(a)$ ) is a root of the polynomial  $a_{-n} \in K[H]$  and  $\text{im}(a_{K[x]_{\leq d}}) + \sum_{j=0}^{\min\{d,n\}} Ka' * x^{d-j} = K[x]_{\leq d}$ ; in this case there is the short exact sequence of finite dimensional vector spaces*

$$0 \rightarrow \ker(a_{K[x]_{\leq d}}) \rightarrow \ker(a_{K[x]}) \rightarrow \ker(\delta) \rightarrow 0$$

where  $\delta = a' \partial^n : \bigoplus_{i=d+1}^{d+n} Kx^i \rightarrow K[x]_{\leq d} / aK[x]_{\leq d}$ , and therefore

$$\ker(a_{K[x]}) = \ker(a_{K[x]_{\leq d}}) \bigoplus_{i=1}^s K(v_i - u_i)$$

where  $\{v_1, \dots, v_s\}$  is a basis for the kernel  $\ker(\delta)$  of  $\delta$  and  $u_1, \dots, u_s \in K[x]_{\leq d}$  be any elements such that  $a' \partial^n * v_i = a * u_i$ .

If the map  $a_{K[x]}$  is surjective then  $n := \text{ind}_{K[x]}(a) = \dim_K(\ker(a_{K[x]})) \geq 0$ .

*Proof.* Suppose that the map  $a_{K[x]}$  is a surjection. Then  $n := \text{ind}_{K[x]}(a) = \dim_K(\ker(a_{K[x]})) \geq 0$  and, by Proposition 6.1.(1),  $a = \sum_{i \geq n} a_{-i} \partial^i + a_F$  is the canonical form of the element  $a$  where  $a_{-n} \neq 0$ , all  $a_j \in K[H]$ , and  $a_F \in F$ . We can write  $a = a' \partial^n + a_F$  where  $a := a_{-n} + a_{-n-1} \partial + \dots$  is an element of the set  $\mathbb{I}_1 \setminus F$  with  $\text{ind}_{K[x]}(a') = 0$ , by Proposition 6.1.(1).

Suppose that  $a_F = 0$ , i.e.,  $a = a' \partial^n$ . Then  $\ker_{K[x]}(a) \supseteq \ker_{K[x]}(\partial^n) = \langle 1, x, \dots, x^{n-1} \rangle$ , and so  $\ker_{K[x]}(a) = \ker_{K[x]}(\partial^n)$  since  $\dim_K(\ker_{K[x]}(a)) = n$ . The map  $a'_{K[x]}$  must be injective: if  $a'p = 0$  for some polynomial  $0 \neq p \in K[x]$  then  $a' \partial^n \int^n p = 0$ ; on the one hand  $\int^n p \in \ker_{K[x]}(a)$  but on the other hand  $\int^n p \notin \langle 1, x, \dots, x^{n-1} \rangle = \ker_{K[x]}(a)$  since  $\deg_x(\int^n p) \geq n$ , a contradiction. By Theorem 6.2.(3), the map  $a'_{K[x]}$  is a bijection since  $\text{ind}_{K[x]}(a') = 0$ . By Theorem 6.2.(1), none of the natural numbers  $j \geq 1$  is a root of the polynomial  $a_{-n}$ . Conversely, suppose that  $a = a' \partial^n$  and none of the natural numbers  $j \geq 1$  is a root of the polynomial  $a_{-n}$ . By Theorem 6.2.(1), the map  $a'_{K[x]}$  is a bijection, then the map  $a_{K[x]}$  is a surjection. This finishes the proof of the theorem in the case when  $a_F = 0$ .

Suppose that  $a_F \neq 0$ . Recall that  $d = \deg_F(a)$ . Suppose, for a moment, that the map  $a_{K[x]}$  is not necessarily a surjection. Applying the Snake Lemma to the commutative diagram of the short exact sequence of vector spaces (where  $V = K[x]_{\leq d}$  and  $U = K[x]/V$ )

$$\begin{array}{ccccccc} 0 & \longrightarrow & V & \longrightarrow & K[x] & \longrightarrow & U \longrightarrow 0 \\ & & \downarrow a_V & & \downarrow a_{K[x]} & & \downarrow a_U \\ 0 & \longrightarrow & V & \longrightarrow & K[x] & \longrightarrow & U \longrightarrow 0 \end{array}$$

we obtain the long exact sequence of vector spaces

$$0 \rightarrow \ker(a_V) \rightarrow \ker(a_{K[x]}) \rightarrow \ker(a_U) \xrightarrow{\delta} \text{coker}(a_V) \rightarrow \text{coker}(a_{K[x]}) \rightarrow \text{coker}(a_U) \rightarrow 0. \quad (31)$$

Therefore, the map  $a_{K[x]}$  is surjective iff the maps  $\delta$  and  $a_U$  are surjective. Clearly, for the element  $a = a' \partial^n + a_F$ ,  $a_U = (a' \partial^n)_U$ . Using the same argument as in the case (a) we conclude that the map  $a_U$  is surjective iff  $a'|_U$  is an isomorphism iff  $a_{-n}|_U$  is an isomorphism iff  $a_{-n}(H) * x^i = a_{-n}(i+1)x^i \neq 0$  for  $i \geq d+1$  iff none of the natural numbers  $i \geq d+2$  is a root of the polynomial  $a_{-n}(H)$ ; and in this case  $\ker(a_U) = \ker(\partial_U^n) = \bigoplus_{i=d+1}^{d+n} Kx^i$ . Let us give more details in the proof of the only non-obvious step above: 'if  $a_U$  is surjective then  $a'|_U$  is an isomorphism.' Notice that the vector space  $U$  is invariant under the action of the elements  $a'$  and  $\partial^n$ ,  $\ker_U(\partial^n) = \bigoplus_{i=d+1}^{d+n} Kx^i \subseteq \ker_U(a' \partial^n)$ , and  $\partial|_U$  is a surjection. Since

$$\ker_U(a' \partial^n) = \text{ind}_U(a' \partial^n) = \text{ind}_U(a') + \text{ind}_U(\partial^n) = 0 + n = n,$$

we must have  $\ker_U(a' \partial^n) = \ker_U(\partial^n)$ . The map  $a'_U$  must be injective; if  $a'u = 0$  for some element  $0 \neq u \in U$  then  $a' \partial^n \int^n u = 0$ ; on the one hand  $\int^n u \in \ker_{K[x]} = \bigoplus_{i=d+1}^{d+n} Kx^i$  but on the other hand  $\int^n u \notin \langle x^{d+1}, x^{d+2}, \dots, x^{d+n} \rangle$ , a contradiction. Therefore,  $a'_U$  is an injection, and so  $a'_U$  is a bijection. Notice that, for all  $i = d+1, \dots, d+n$ ,

$$\delta(x^i) = a * x^i + \text{im}(a_V) = \begin{cases} i(i-1) \cdots (i-n+1) a' * x^{i-n} & \text{if } i \geq n, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the map  $\delta$  is surjective iff  $\text{im}(a_V) + \sum_{j=0}^{\min\{d,n\}} K a' * x^{d-j} = V$ . This finishes the proof of 'iff' part of case 2 of the theorem.

To finish the proof of case 2 suppose that the map  $a_{K[x]}$  is surjective. By (31), there is the short exact sequence of vector spaces  $0 \rightarrow \ker(a_V) \rightarrow \ker(a_{K[x]}) \rightarrow \ker(\delta) \rightarrow 0$  where  $\delta = a' \partial^n : \bigoplus_{i=d+1}^{d+n} Kx^i \rightarrow V/aV$ . Then  $\ker(a_{K[x]}) = \ker(a_V) \oplus \bigoplus_{i=1}^s K(v_i - u_i)$  where  $\{v_1, \dots, v_s\}$  is a basis for the kernel  $\ker(\delta)$  of  $\delta$  and  $u_1, \dots, u_s \in V$  be any elements such that  $a' \partial^n * v_i = a * u_i$ .  $\square$

**Corollary 6.4** *Let an element  $a \in \mathbb{I}_1$  be such that the map  $a_{K[x]}$  is surjective. Then the element  $a$  can be uniquely written as the product  $a = \partial^n a'$  where  $n = \dim_K(\ker(a_{K[x]}))$ ,  $a' = \sum_{i \geq 0} a'_i \partial^i + f$ ,  $a'_0 \neq 0$ , all  $a'_i \in K[H]$ ,  $f = \sum_{i,j \geq 0} \lambda_{i+n,j} e_{i+n,j}$ ,  $\lambda_{i+n,j} \in K$ .*

*Proof.* The existence and uniqueness follow from Theorem 6.3 and the equalities  $\alpha(H) \cdot \partial^n = \partial^n \cdot \alpha(H+n)$ , for all elements  $\alpha(H) \in K[H]$ , and  $\partial^n \cdot e_{ij} = e_{i-n,j}$ .  $\square$

**Classification of elements  $a \in \mathbb{I}_1$  such that  $a_{K[x]}$  is an injection.**

**Lemma 6.5** *Let  $a = \sum_{i \geq 1} a_{-i} \partial^i + a_0 + f \in \mathbb{I}_1$  where all  $a_j \in K[H]$ ,  $a_0 \neq 0$ , and  $f \in F$ . Then  $\ker_{K[x]}(a \cdot) = \ker_{K[x]_{\leq m}}(a \cdot)$  and  $\text{coker}_{K[x]}(a \cdot) \simeq \text{coker}_{K[x]_{\leq m}}(a \cdot)$  where  $m = m(a) = \max\{\deg_F(a), s\}$  and  $s = \max\{j \in \mathbb{N} \mid j+1 \text{ is a root of the polynomial } a_0 \in K[H]\}$  or  $s := -1$  if there is no such a root  $j+1$ .*

*Proof.* Notice that  $K[x]_{\leq m}$  is a  $K[a]$ -submodule of  $K[x]$  such that the map  $a|_{K[x]/K[x]_{\leq m}}$  is a bijection by the choice of  $m$ . Then (31) yields the result.  $\square$

Let  $a \in \mathbb{I}_1$  be such that  $n = \max\{i > 0 \mid a_i \neq 0\} \neq 0$ , see (5). By Proposition 6.1.(1),  $n = -\text{ind}_{K[x]}(a) > 0$ . Then the element  $a$  is a *unique* sum (Proposition 6.1.(2b))

$$a = \left( \sum_{i \geq 1} b_{-i} \partial^i + b_0 + f \right) \int^n \quad (32)$$

where all  $b_j \in K[H]$ ,  $b_0 \neq 0$ ,  $f \in \sum_{i,j \geq 0} K e_{i,j+n}$ .

**Theorem 6.6** *Let  $a \in \mathbb{I}_1$ . Then the map  $a_{K[x]}$  is an injection iff  $a = (\sum_{i \geq 1} a_{-i} \partial^i + a_0 + f) \int^n$  where  $n \geq 0$ , all  $a_j \in K[H]$ ,  $a_0 \neq 0$ ,  $f \in \sum_{i,j \geq 0} K e_{i,j+n}$  (the presentation for  $a$  is unique), and*

1. *when  $n = 0$  the map  $a_{K[x]}$  is a bijection (see Theorem 6.2.(1)); and*

2. when  $n \geq 1$ ,  $\ker_{K[x]_{\leq m}}(a' \cdot) \cap x^n K[x] = 0$  where  $a' := \sum_{i \geq 1} a_{-i} \partial^i + a_0 + f$  and  $m = m(a')$  is as in Lemma 6.5; in this case there is the short exact sequence of finite dimensional vector spaces

$$0 \rightarrow \ker_{K[x]_{\leq m}}(a' \cdot) \rightarrow K[x]/(x^n) \rightarrow \operatorname{coker}(a_{K[x]}) \rightarrow K[x]_{\leq m}/a' * K[x]_{\leq m} \rightarrow 0.$$

If the map  $a'_{K[H]}$  is a bijection then  $n = -\operatorname{ind}_{K[x]}(a) = \dim_K(\operatorname{coker}(a_{K[x]}))$ .

*Proof.* Suppose that the map  $a_{K[x]}$  is injective. Then  $n := -\operatorname{ind}_{K[x]}(a) = \dim_K(\operatorname{coker}(a_{K[x]})) \geq 0$  and, by Proposition 6.1.(1) and (32), the element  $a$  is the unique sum  $a = (\sum_{i \geq 1} a_{-i} \partial^i + a_0 + f) \int^n = a' \int^n$  where all  $a_j \in K[H]$ ,  $a_0 \neq 0$  and  $f \in \sum_{i,j \in \mathbb{N}} K e_{i,j+n}$ . To finish the proof of the theorem we have to show that for each element  $a$  that admits such a presentation,  $a = a' \int^n$  for some  $n \in \mathbb{N}$ , the map  $a_{K[x]}$  is an injection iff the conditions 1 and 2 hold. If  $n = 0$  then this follows from Theorem 6.2.(1,3). Suppose that  $n \geq 1$ . Then the map  $a_{K[x]}$  is an injection iff  $\ker_{K[x]}(a' \cdot) \cap \operatorname{im}_{K[x]}(\int^n \cdot) = 0$  iff  $\ker_{K[x]_{\leq m}}(a' \cdot) \cap (x^n) = 0$ , by Lemma 6.5. This completes the ‘iff’ part of statement 2. It remains to prove existence of the long exact sequence in statement 2. So, let  $n \geq 1$  and the map  $a_{K[x]}$  is injective. Applying (9) for the product  $a = a' \int^n$ , yields the long exact sequence of vector spaces

$$0 \rightarrow \ker_{K[x]}(a' \cdot) \rightarrow \operatorname{coker}_{K[x]}(\int^n) \rightarrow \operatorname{coker}(a_{K[x]}) \rightarrow \operatorname{coker}_{K[x]}(a' \cdot) \rightarrow 0.$$

Notice that  $\ker_{K[x]}(a' \cdot) = \ker_{K[x]_{\leq m}}(a' \cdot)$ ,  $\operatorname{coker}_{K[x]}(a' \cdot) \simeq \operatorname{coker}_{K[x]_{\leq m}}(a' \cdot)$  (Lemma 6.5) and  $\operatorname{coker}_{K[x]}(\int^n \cdot) = K[x]/(x^n)$ . The proof is complete.  $\square$

In general, for a linear map acting in an infinite dimensional space it is not easy to find its cokernel. The next several results make this problem finite dimensional for integro-differential operators.

**Corollary 6.7** *Let an element  $a \in \mathbb{I}_1$  be such that the map  $a_{K[x]}$  is an injection with  $n = \dim_K(\operatorname{coker}(a_{K[x]})) \geq 1$ , and  $a = \int^n a' + f$  be the unique sum where  $a' = \sum_{i \geq 1} a_{-i} \partial^i + a_0$ , all  $a_j \in K[H]$ ,  $a_0 \neq 0$ , and  $f \in F$  (Theorem 6.6). Then the set  $\{1, x, \dots, x^{n-1}\}$  is a basis for  $\operatorname{coker}(a_{K[x]})$  iff the map  $(a' + \partial^n f)_{K[x]}$  is a bijection (see Theorem 6.2.(1) for the classification of bijections).*

*Remark.* The existence and uniqueness of the presentation  $a = \int^n a' + f$  follows from Theorem 6.6 and (3).

*Proof.* ( $\Rightarrow$ ) Suppose that the set  $\{1, x, \dots, x^{n-1}\} = \ker_{K[x]}(\partial^n \cdot)$  is a basis for the cokernel of the map  $a_{K[x]}$ . Then  $K[x] = \ker_{K[x]}(\partial^n \cdot) \oplus \operatorname{im}(a_{K[x]})$ , and so the map  $(\partial^n a)_{K[x]} = (a' + \partial^n f)_{K[x]}$  is an injection, hence it is a bijection, by Theorem 6.2.(3).

( $\Leftarrow$ ) Suppose that the map  $(\partial^n a)_{K[x]} = (a' + \partial^n f)_{K[x]}$  is a bijection. Then  $\ker_{K[x]}(\partial^n \cdot) \cap \operatorname{im}(a_{K[x]}) = 0$ , and so  $\ker_{K[x]}(\partial^n \cdot) \oplus \operatorname{im}(a_{K[x]}) = K[x]$  since  $\dim_K(\operatorname{coker}(a_{K[x]})) = n$  and  $n = \dim_K(\ker_{K[x]}(\partial^n \cdot))$ . Therefore, the set  $\{1, x, \dots, x^{n-1}\}$  is a basis for  $\operatorname{coker}(a_{K[x]})$ .  $\square$

**Proposition 6.8** *Let  $V$  be a nonzero finite dimensional subspace of  $K[x]$  of dimension  $n$ . Then there exists a unit  $s \in (1+F)^*$  such that  $V = \ker(s \partial^n s^{-1})_{K[x]}$  and  $K[x] = V \oplus \operatorname{im}(s \int^n s^{-1})_{K[x]}$ , i.e.,  $V \simeq \operatorname{coker}(s \int^n s^{-1})_{K[x]}$ . In particular,  $s \partial^n s^{-1} = \partial^n + g$  and  $s \int^n s^{-1} = \int^n + f$  for some elements  $g, f \in F$ .*

*Proof.* If  $V = \langle 1, x, \dots, x^{n-1} \rangle = \ker(\partial^n)_{K[x]}$  then take  $s = 1$  as  $K[x] = \ker(\partial^n)_{K[x]} \oplus \operatorname{im}(\int^n)_{K[x]}$ . In the general case, fix a natural number  $m \geq n$  and subspaces  $U, V, W \subseteq K[x]_{\leq m}$  such that  $K[x]_{\leq m} = V \oplus U = \ker(\partial^n)_{K[x]} \oplus W$ . Since  $(1+F)^* = \operatorname{GL}_\infty(K)$ , we can find an element

$s$  of  $(1 + F)^*$  such that  $s^{-1}(V) = \ker(\partial^n)_{K[x]}$ ,  $s^{-1}(U) = W$  and  $s(u) = u$  for all elements  $u \in K[x]_{>m} := \bigoplus_{i>m} Kx^i$ . Then the element  $s$  satisfies the conditions of the proposition:

$$K[x] = s \ker(\partial^n)_{K[x]} \bigoplus s \operatorname{im}(\int^n)_{K[x]} = \ker(s \partial^n s^{-1})_{K[x]} \bigoplus \operatorname{im}(s \int^n s^{-1}) = V \bigoplus \operatorname{im}(s \int^n s^{-1}).$$

In particular,  $s = 1 + h$  and  $s^{-1} = 1 + h'$  for some elements  $h, h' \in F$ , and so  $s \partial^n s^{-1} = \partial^n + g$  and  $s \int^n s^{-1} = \int^n + f$  for some elements  $g, f \in F$ .  $\square$

**Corollary 6.9** *Let an element  $a \in \mathbb{I}_1$  be such that the map  $a_{K[x]}$  is an injection with  $n = \dim_K(\operatorname{coker}(a_{K[x]})) \geq 1$ ; then  $a = \int^n a' + f$  is the unique sum where  $a' = \sum_{i \geq 1} a_{-i} \partial^i + a_0$ , all  $a_j \in K[H]$ ,  $a_0 \neq 0$ , and  $f \in F$  (Theorem 6.6). Let  $g \in F$  be such that the map  $(\partial^n + g)_{K[x]}$  is a surjection. Then  $\ker_{K[x]}(\partial^n + g) \simeq \operatorname{coker}(a_{K[x]})$  (i.e.,  $K[x] = \ker_{K[x]}(\partial^n + g) \bigoplus \operatorname{im}(a_{K[x]})$ ) iff the map  $((\partial^n + g)a)_{K[x]} = (a' + h)_{K[x]}$  is a bijection (see Theorem 6.2.(1) for the classification of bijections) where  $h := \partial^n f + ga \in F$ .*

*Proof.*  $(\Rightarrow)$  Suppose that  $K[x] = \ker_{K[x]}(\partial^n + g) \bigoplus \operatorname{im}(a_{K[x]})$ . Then the map  $((\partial^n + g)a)_{K[x]} = (a' + h)_{K[x]}$  is a injection, hence it is a bijection, by Theorem 6.2.(3).

$(\Leftarrow)$  Suppose that the map  $((\partial^n + g)a)_{K[x]} = (a' + h)_{K[x]}$  is a bijection. Then  $\ker_{K[x]}(\partial^n + g) \cap \operatorname{im}(a_{K[x]}) = 0$ , and so  $\ker_{K[x]}(\partial^n + g) \bigoplus \operatorname{im}(a_{K[x]}) = K[x]$  since  $n = -\deg_{\partial^{-1}}(\partial^n + g + F) = \operatorname{ind}_{K[x]}(\partial^n + g) = \dim_K(\ker_{K[x]}(\partial^n + g))$ , by Proposition 6.1.(1), and  $n = \dim_K(\operatorname{coker}(a_{K[x]}))$ .  $\square$

Corollary 6.9 is an effective tool in finding a basis for the cokernel of an injection  $a_{K[x]}$ .

*Example.* Let  $a = \partial + f$ . Then  $a_{K[x]}$  is an injection and the map  $\partial_{K[x]}$  is a surjection such that  $(\partial a)_{K[x]} = (\partial^2 + 1)_{K[x]} \in \operatorname{Aut}_K(K[x])$  is a bijection since  $(1 + \partial^2)^{-1}_{K[x]} = \sum_{i \geq 0} (-1)^i \partial_{K[x]}^{2i}$ . By Corollary 6.9,  $\operatorname{coker}_{K[x]}(\partial + f) \simeq \ker_{K[x]}(\partial) = K$ . Notice that  $(1 + \partial^2)^{-1} \notin \mathbb{I}_1^*$  where  $\mathbb{I}_1^*$  is the group of units of the algebra  $\mathbb{I}_1$ .

**Lemma 6.10** 1. *For each element  $a \in \mathbb{I}_1 \setminus F$ , there exists an idempotent  $f \in F$  such that  $\ker(a_{K[x]}) = \operatorname{im}(f_{K[x]})$ .*

2. *Let  $a \in \mathbb{I}_1 \setminus F$ . Then there exists an element  $g \in F$  such that  $\operatorname{im}(a_{K[x]}) = \ker(g_{K[x]})$  iff there exists a natural number  $d \geq 0$  such that  $x^{d+1}K[x] \subseteq \operatorname{im}(a_{K[x]})$  and*

$$\operatorname{codim}_{K[x]_{\leq d}}(K[x]_{\leq d} \bigcap \operatorname{im}(a_{K[x]})) = \dim_K(\operatorname{coker}(a_{K[x]})).$$

*In this case, the element  $g$  can be chosen to be an idempotent.*

*Proof.* 1. Since  $a \in \mathbb{I}_1 \setminus F$ , the kernel of the linear map  $a_{K[x]}$  is a finite dimensional vector space (Theorem 3.1.(1)), and so  $\ker(a_{K[x]}) \subseteq K[x]_{\leq m}$  for some natural number  $m$ . Then  $K[x]_{\leq m} = \ker(a_{K[x]}) \bigoplus V$  for some subspace  $V$  of  $K[x]_{\leq m}$ . If  $f$  is the projection onto the direct summand  $\ker(a_{K[x]})$  of  $K[x]_{\leq m}$  extended by zero on  $K[x]_{>m}$  then  $f^2 = f$ ,  $f \in F$  and  $\ker(a_{K[x]}) = \operatorname{im}(f_{K[x]})$ .

2.  $(\Rightarrow)$  Suppose that there exists an element  $g \in F$  such that  $\operatorname{im}(a_{K[x]}) = \ker(g_{K[x]})$ . Let  $d = \deg_F(g)$ . Then  $K[x] = K[x]_{\leq d} \bigoplus (x^{d+1})$  and  $\ker(g_{K[x]}) = \ker(g_{K[x]_{\leq d}}) \bigoplus (x^{d+1})$ . Therefore  $(x^{d+1}) \subseteq \operatorname{im}(a_{K[x]})$  and  $\operatorname{codim}_{K[x]_{\leq d}}(K[x]_{\leq d} \bigcap \operatorname{im}(a_{K[x]})) = \operatorname{codim}_{K[x]_{\leq d}}(\ker(g_{K[x]_{\leq d}})) = \operatorname{codim}_{K[x]}(\ker(g_{K[x]})) = \dim_K(\operatorname{coker}(a_{K[x]}))$ .

$(\Leftarrow)$  Suppose that  $(x^{d+1}) \subseteq \operatorname{im}(a_{K[x]})$  and

$$\operatorname{codim}_{K[x]_{\leq d}}(K[x]_{\leq d} \bigcap \operatorname{im}(a_{K[x]})) = \dim_K(\operatorname{coker}(a_{K[x]})) =: n.$$

Then  $K[x]_{\leq d} = V \bigoplus (K[x]_{\leq d} \bigcap \operatorname{im}(a_{K[x]}))$  for some subspace  $V$  of  $K[x]_{\leq d}$  with  $\dim_K(V) = n$ . Then  $\operatorname{im}(a_{K[x]}) = K[x]_{\leq d} \bigcap \operatorname{im}(a_{K[x]}) \bigoplus (x^{d+1})$ . It suffices to take  $g$  which is the projection map  $K[x]_{\leq d} \rightarrow V$  extended by zero on the ideal  $(x^{d+1})$ .  $\square$

*Non-Example.* The conditions of statement 2 are very restrictive. In particular, not for every element  $a \in \mathbb{I}_1 \setminus F$  there exists an element  $g$  such that  $\text{im}(a_{K[x]}) = \ker(g_{K[x]})$ , eg,  $a = \partial + \int$  since  $x^{2n} \notin \text{im}(\partial + \int)$  for all  $n \geq 0$ .

*Proof.* We have to show that there is no polynomial solution  $u$  to the equation  $(\partial + \int) * u = x^{[2n]}$ . Note that

$$\partial + \int = \partial \int (\partial + \int) = \partial(1 - e_{00} + \int^2) = \partial(1 - e_{00})(1 + \int^2).$$

Then,  $(1 - e_{00})(1 + \int^2) * u = x^{[2n+1]} + C$  for some constant  $C$  necessarily  $C = 0$  as  $\text{im}(1 - e_{00})_{K[x]} = (x) \ni x^{[2n+1]}$ . We can write  $u = \lambda + v$  for some  $\lambda \in K$  and  $v \in (x)$ . The linear maps  $1 - e_{00}$  and  $1 + \int^2$  acting in  $K[x]$  respect the ideal  $(x)$ . By taking the equality  $(1 - e_{00})(1 + \int^2) * u = x^{[2n+1]}$  modulo  $(x)$  yields  $(1 - e_{00})\lambda = 0$  and so  $\lambda = 0$ . Since the  $(1 - e_{00})_{K[x]}$  is the projection onto the ideal  $(x)$  in the decomposition  $K[x] = K \oplus (x)$ ,  $\lambda = 0$  and  $x^{[2n+1]} \in (x)$ , we can drop  $1 - e_{00}$  in the equation, i.e.,  $(1 + \int^2) * u = x^{[2n+1]}$ . The only solution  $u = \sum_{i \geq 0} (-1)^i \int^{2i} * x^{[2n+1]} \in K[[x]]$  is obviously not a polynomial.  $\square$

**Proposition 6.11** 1. For each element  $a \in \mathbb{I}_1 \setminus F$  with  $n := \dim_K(\text{coker}(a_{K[x]}))$ , there exists an element  $\partial^n + f$  for some  $f \in F$  (resp.  $s \in (1 + F)^*$ ) such that the map  $(\partial^n + f)a_{K[x]}$  (resp.  $s\partial^n s^{-1}a_{K[x]}$ ) is a surjection. In this case,  $\ker((\partial^n + f)a_{K[x]}) = \ker(a_{K[x]})$  (resp.  $\ker(s\partial^n s^{-1}a_{K[x]}) = \ker(a_{K[x]})$ ).

2. For each element  $a \in \mathbb{I}_1 \setminus F$  with  $n := \dim_K(\ker(a_{K[x]}))$ , there exists an element  $\int^n + g$  for some  $g \in F$  (resp.  $s \in (1 + F)^*$ ) such that the map  $a(\int^n + g)_{K[x]}$  (resp.  $as\int^n s^{-1}_{K[x]}$ ) is an injection. In this case,  $\text{im}(a(\int^n + g)_{K[x]}) = \text{im}(a_{K[x]})$  (resp.  $\text{im}(as\int^n s^{-1}_{K[x]}) = \text{im}(a_{K[x]})$ ).

3. For each element  $a \in \mathbb{I}_1 \setminus F$  with  $m := \dim_K(\ker(a_{K[x]}))$  and  $n := \dim_K(\text{coker}(a_{K[x]}))$ , there exist elements  $\partial^n + f$  and  $\int^m + g$  for some  $f, g \in F$  (resp.  $s, t \in (1 + F)^*$ ) such that the map  $(\partial^n + f)a(\int^m + g)_{K[x]}$  (resp.  $s\partial^n s^{-1}at\int^m t^{-1}_{K[x]}$ ) is a bijection.

*Proof.* 1. Notice that  $K[x] = V \oplus \text{im}(a_{K[x]})$  for some  $n$ -dimensional subspace  $V$  of  $K[x]$ . By Proposition 6.8, there exists a unit  $s \in (1 + F)^*$  such that  $V = \ker(s\partial^n s^{-1}_{K[x]})$  and  $s\partial^n s^{-1} = \partial^n + f$  for some  $f \in F$ . Then the map  $(s\partial^n s^{-1})a_{K[x]}$  is surjective since  $K[x] = s\partial^n s^{-1} * K[x] = s\partial^n s^{-1}(\ker(s\partial^n s^{-1}) \oplus \text{im}(a_{K[x]})) = \text{im}((s\partial^n s^{-1})a_{K[x]})$ . Then, by Lemma 3.5,

$$\begin{aligned} \dim_K \ker((\partial^n + f)a_{K[x]}) &= \text{ind}_{K[x]}((\partial^n + f)a) = \text{ind}_{K[x]}(\partial^n + f) + \text{ind}_{K[x]}(a) \\ &= \text{ind}_{K[x]}(\partial^n) + \text{ind}_{K[x]}(a) = n + \dim_K \ker(a_{K[x]}) - n = \dim_K \ker(a_{K[x]}). \end{aligned}$$

Therefore,  $\ker((\partial^n + f)a_{K[x]}) = \ker(a_{K[x]})$  since  $\ker((\partial^n + f)a_{K[x]}) \supseteq \ker(a_{K[x]})$ .

2. Since  $a \notin F$ , the kernel  $V$  of the linear map  $a_{K[x]}$  is finite dimensional (Theorem 3.1). By Proposition 6.8, there exists a unit  $s \in (1 + F)^*$  such that  $K[x] = V \oplus \text{im}(s\int^n s^{-1}_{K[x]})$  and  $s\int^n s^{-1} = \int^n + g$  for some element  $g \in F$ . It follows that the map  $a(\int^n + g)_{K[x]}$  is an injection. Then, by Lemma 3.5,

$$\begin{aligned} -\dim_K \text{coker}(a(\int^n + g)_{K[x]}) &= \text{ind}_{K[x]}(a(\int^n + g)) = \text{ind}_{K[x]}(a) + \text{ind}_{K[x]}(\int^n + g) \\ &= \text{ind}_{K[x]}(a) + \text{ind}_{K[x]}(\int^n) = n - \dim_K \text{coker}(a_{K[x]}) - n \\ &= -\dim_K \text{coker}(a_{K[x]}). \end{aligned}$$

Therefore, the natural inclusion  $\text{im}(a(\int^n + g)_{K[x]}) \subseteq \text{im}(a_{K[x]})$  is an equality.

3. By statement 1, there exists an element  $\partial^n + f$  for some  $f \in F$  (resp.  $s \in (1 + F)^*$ ) such that the map  $a' := (\partial^n + f)a_{K[x]}$  (resp.  $a' := s\partial^n s^{-1}a_{K[x]}$ ) is a surjection with  $\ker(a'_{K[x]}) = \ker(a_{K[x]})$ . Then, by statement 2, for the element  $a'$ , there exists an element  $\int^m + g$  for some  $g \in F$  (resp.

$t \in (1 + F)^*$  such that the map  $a'' := a'(\int^m + g)_{K[x]}$  (resp.  $a'' := a't \int^m t_{K[x]}^{-1}$ ) is an injection with  $\text{im}(a''_{K[x]}) = \text{im}(a'_{K[x]}) = K[x]$ , i.e., the map  $a''_{K[x]}$  is a bijection.  $\square$

*Example.* Let  $a = \partial + \int$ . We know already that  $a_{K[x]}$  is an injection with  $\dim_K(\text{coker}(a_{K[x]})) = 1$ . Then  $\partial a = \partial^2 + 1$  and  $(1 + \partial^2)_{K[x]}$  is a bijection.

The next proposition is useful in proving that various Ext's and Tor's are finite dimensional vector spaces or not.

**Proposition 6.12** *Let  $a, b \in \mathbb{I}_1$ .*

1. *If  $a, b \notin F$  then the vector spaces  $\ker_{\ker_{\mathbb{I}_1}(\cdot b)}(a \cdot)$ ,  $\text{coker}_{\ker_{\mathbb{I}_1}(\cdot b)}(a \cdot)$ ,  $\ker_{\text{coker}_{\mathbb{I}_1}(\cdot b)}(a \cdot)$ ,  $\text{coker}_{\text{coker}_{\mathbb{I}_1}(\cdot b)}(a \cdot)$  are finite dimensional.*
2. *If  $a \notin F$  and  $b \in F$  then*

- (a)  $\ker_{\ker_F(\cdot b)}(a \cdot) = \ker_{\ker_{\mathbb{I}_1}(\cdot b)}(a \cdot)$  and  $\dim_K(\ker_{\ker_{\mathbb{I}_1}(\cdot b)}(a \cdot)) = \begin{cases} 0 & \text{if } \ker_{K[x]}(a \cdot) = 0, \\ \infty & \text{otherwise.} \end{cases}$
- (b) *The sequence  $0 \rightarrow \text{coker}_{\ker_F(\cdot b)}(a \cdot) \rightarrow \text{coker}_{\ker_{\mathbb{I}_1}(\cdot b)}(a \cdot) \rightarrow \text{coker}_{B_1}(a \cdot) \rightarrow 0$  is exact and*  
 $\dim_K(\text{coker}_{\ker_{\mathbb{I}_1}(\cdot b)}(a \cdot)) = \begin{cases} 0 & \text{if } a = \lambda \partial^i + f : K[x] \rightarrow K[x] \text{ is a surjection,} \\ \infty & \text{otherwise,} \end{cases}$   
*for some  $\lambda \in K^*$ ,  $i \geq 0$  and  $f \in F$ .*
- (c)  $\ker_{\text{coker}_{\mathbb{I}_1}(\cdot b)}(a \cdot) \simeq \ker_{\text{coker}_F(\cdot b)}(a \cdot)$  and  $\dim_K(\ker_{\text{coker}_{\mathbb{I}_1}(\cdot b)}(a \cdot)) = \begin{cases} 0 & \text{if } a_{K[x]} \text{ is injective,} \\ \infty & \text{otherwise.} \end{cases}$
- (d) *The sequence  $0 \rightarrow \text{coker}_{\text{coker}_F(\cdot b)}(a \cdot) \rightarrow \text{coker}_{\text{coker}_{\mathbb{I}_1}(\cdot b)}(a \cdot) \rightarrow B_1/aB_1 \rightarrow 0$  is exact and*  
 $\dim_K(\text{coker}_{\text{coker}_{\mathbb{I}_1}(\cdot b)}(a \cdot)) = \begin{cases} 0 & \text{if } a = \lambda \partial^i + f, a_{K[x]} \text{ is surjective,} \\ \infty & \text{otherwise,} \end{cases}$   
*for some  $\lambda \in K^*$ ,  $i \geq 0$  and  $f \in F$ .*

3. *If  $a \in F$  and  $b \notin F$  then*

- (a)  $\ker_{\ker_{\mathbb{I}_1}(\cdot b)}(a \cdot) = \ker_{\ker_F(\cdot b)}(a \cdot)$  and  $\dim_K(\ker_{\ker_{\mathbb{I}_1}(\cdot b)}(a \cdot)) = \begin{cases} \infty & \text{if } \ker_{\mathbb{I}_1}(\cdot b) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$   
 $\ker_{\mathbb{I}_1}(\cdot b) = 0 \Leftrightarrow (\cdot b)_{\mathbb{I}_1} \text{ is an injection} \Leftrightarrow \cdot b : \mathbb{I}_1 / \int \mathbb{I}_1 \rightarrow \mathbb{I}_1 / \int \mathbb{I}_1 \text{ is an injection} \Leftrightarrow$   
 $b^* \cdot : K[x] \rightarrow K[x] \text{ is an injection.}$
- (b)  $\text{coker}_{\ker_{\mathbb{I}_1}(\cdot b)}(a \cdot) \simeq \text{coker}_{\ker_F(\cdot b)}(a \cdot)$  and  $\dim_K(\text{coker}_{\ker_{\mathbb{I}_1}(\cdot b)}(a \cdot)) = \begin{cases} \infty & \text{if } \ker_{\mathbb{I}_1}(\cdot b) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$
- (c)  $\dim_K(\ker_{\text{coker}_{\mathbb{I}_1}(\cdot b)}(a \cdot)) = \begin{cases} 0 & \text{if } b = \lambda \int^i + f, (b^* \cdot)_{K[x]} \text{ is surjective,} \\ \infty & \text{otherwise,} \end{cases}$   
*where  $0 \neq \lambda \in K$ ,  $i \geq 0$  and  $f \in F$ .*
- (d)  $\dim_K(\text{coker}_{\text{coker}_{\mathbb{I}_1}(\cdot b)}(a \cdot)) = \begin{cases} 0 & \text{if } b = \lambda \int^i + f, (b^* \cdot)_{K[x]} \text{ is surjective,} \\ \infty & \text{otherwise,} \end{cases}$   
*where  $0 \neq \lambda \in K$ ,  $i \geq 0$  and  $f \in F$ .*

4. *If  $a, b \in F$  then the vector spaces  $\ker_{\ker_{\mathbb{I}_1}(\cdot b)}(a \cdot)$ ,  $\text{coker}_{\ker_{\mathbb{I}_1}(\cdot b)}(a \cdot)$ ,  $\ker_{\text{coker}_{\mathbb{I}_1}(\cdot b)}(a \cdot)$ ,  $\text{coker}_{\text{coker}_{\mathbb{I}_1}(\cdot b)}(a \cdot)$  are infinite dimensional.*



*Remark.* Earlier, necessary and sufficient conditions were given for the map  $a : K[x] \rightarrow K[x]$  to be bijective (Theorem 6.2), surjective (Theorem 6.3) or injective (Theorem 6.6).

*Proof.* 1. Since  $b \notin F$ , the left  $\mathbb{I}_1$ -modules  $\ker_{\mathbb{I}_1}(\cdot b)$  and  $\text{coker}_{\mathbb{I}_1}(\cdot b)$  have finite length (Theorem 3.6). Then, by Theorem 3.1, all four vector spaces are finite dimensional.

2(a,b). Since  $b \in F$ ,  $\ker_{B_1}(\cdot b) = \ker_{B_1}(0) = B_1$ . Since  $a \notin F$ ,  $\ker_{\ker_{B_1}(\cdot b)}(a \cdot) = \ker_{B_1}(a \cdot) = 0$ . Therefore, the long exact sequence in Lemma 4.1.(2a) brakes down into two short exact sequences  $\ker_{\ker_F(\cdot b)}(a \cdot) = \ker_{\ker_{\mathbb{I}_1}(\cdot b)}(a \cdot)$  and  $0 \rightarrow \text{coker}_{\ker_F(\cdot b)}(a \cdot) \rightarrow \text{coker}_{\ker_{\mathbb{I}_1}(\cdot b)}(a \cdot) \rightarrow \text{coker}_{B_1}(a \cdot) \rightarrow 0$ . Since  $b \in F$ ,  ${}_{\mathbb{I}_1}\ker_F(\cdot b) \simeq K[x]^{(\mathbb{N})}$ , a direct sum of countably many copies of the left  $\mathbb{I}_1$ -module  $K[x]$ , then statement (a) follows. Using the short exact sequence, we see that the vector space  $\text{coker}_{\ker_{\mathbb{I}_1}(\cdot b)}(a \cdot)$  is finite dimensional iff so are the vector spaces  $\text{coker}_{\ker_F(\cdot b)}(a \cdot)$  and  $\text{coker}_{B_1}(a \cdot)$ . Since  $\ker_F(\cdot b) \simeq K[x]^{(\mathbb{N})}$ , the vector space  $\text{coker}_{\ker_F(\cdot b)}(a \cdot)$  is finite dimensional iff it is a zero space iff the map  $a_{K[x]}$  is surjective. Since  $a \notin F$ , the vector space  $\text{coker}_{\ker_{B_1}(\cdot b)}(a \cdot) = \text{coker}_{B_1}(a \cdot)$  is finite dimensional iff  $a + F$  is a unit of the algebra  $B_1$  iff  $a + F = \lambda \partial^i \in B_1$  where  $\lambda \in K^*$  and  $i \in \mathbb{Z}$  iff  $\text{coker}_{B_1}(a \cdot) = 0$ . By Theorem 6.3, the vector space  $\text{coker}_{\ker_{\mathbb{I}_1}(\cdot b)}(a \cdot)$  is finite dimensional iff it is a zero space iff  $a = \lambda \partial^i + f$  for some  $\lambda \in K^*$ ,  $i \geq 0$ , and  $f \in F$  such that the map  $(\lambda \partial^i + f)_{K[x]}$  is surjective.

2(c,d). Since  $a \notin F$  and  $b \in F$ , we have  $\text{coker}_{B_1}(\cdot b) = B_1$  and  $\ker_{\text{coker}_{B_1}(\cdot b)}(a \cdot) = 0$ . Therefore, the long exact sequence in Lemma 4.1.(2b) collapses to  $\ker_{\text{coker}_F(\cdot b)}(a \cdot) \simeq U := \ker_{\text{coker}_{\mathbb{I}_1}(\cdot b)}(a \cdot)$  and the short exact sequence

$$0 \rightarrow V_1 := \text{coker}_{\text{coker}_F(\cdot b)}(a \cdot) \rightarrow V := \text{coker}_{\text{coker}_{\mathbb{I}_1}(\cdot b)}(a \cdot) \rightarrow B_1/aB_1 \rightarrow 0.$$

Since  ${}_{\mathbb{I}_1}F_{\mathbb{I}_1} \simeq K[x] \otimes (\mathbb{I}_1/\int \mathbb{I}_1)$  and  $b \in F$ ,  ${}_{\mathbb{I}_1}\text{coker}_F(\cdot b) \simeq K[x]^{(\mathbb{N})}$ . Therefore,

$$\dim_K(U) = \begin{cases} 0 & \text{if } a_{K[x]} \text{ is an injection,} \\ \infty & \text{otherwise.} \end{cases}$$

Notice that  $\dim_K(V) = \dim_K(V_1) + \dim_K(B_1/aB_1)$ .

$$\dim_K(B_1/aB_1) = \begin{cases} 0 & \text{if } a = \lambda \partial^i + f, \lambda \int^i + f, \\ \infty & \text{otherwise,} \end{cases}$$

where  $0 \neq \lambda \in K$ ,  $i \geq 0$  and  $f \in F$ .  $\dim_K(V_1) = \begin{cases} 0 & \text{if } a_{K[x]} \text{ is surjective,} \\ \infty & \text{otherwise.} \end{cases}$

By Theorem 6.3,  $\dim_K(V) = 0$  iff  $a = \lambda \partial^i + f$  and  $a_{K[x]}$  is a surjection where  $0 \neq \lambda \in K$ ,  $i \geq 0$  and  $f \in F$ ; otherwise  $\dim_K(V) = \infty$ .

3(a,b). Notice that  $\ker_{\mathbb{I}_1}(\cdot b) = 0 \Leftrightarrow (\cdot b)_{\mathbb{I}_1}$  is an injection  $\Leftrightarrow (\cdot b)_F$  is an injection (since  $b \notin F$ )  $\Leftrightarrow \cdot b : \mathbb{I}_1/\int \mathbb{I}_1 \rightarrow \mathbb{I}_1/\int \mathbb{I}_1$  is an injection (since  $F_{\mathbb{I}_1} \simeq (\mathbb{I}_1/\int \mathbb{I}_1)^{(\mathbb{N})}$ )  $\Leftrightarrow b^* : K[x] \rightarrow K[x]$  is an injection since  ${}_{\mathbb{I}_1}K[x] \simeq \mathbb{I}_1/\mathbb{I}_1\partial$  and the map  $* : \mathbb{I}_1/\int \mathbb{I}_1 \rightarrow \mathbb{I}_1/\mathbb{I}_1\partial$ ,  $u + \int \mathbb{I}_1 \mapsto u^* + \mathbb{I}_1\partial$ , is a bijection such that  $(cu)^* = u^*c^*$  for all elements  $c \in \mathbb{I}_1$  and  $u \in \mathbb{I}_1/\int \mathbb{I}_1$ .

Since  $b \notin F$ , we have  $\ker_{B_1}(\cdot b) = 0$ , and so the short exact sequence in Theorem 4.1.(2a) yields  $\ker_{\ker_F(\cdot b)}(a \cdot) = \ker_{\ker_{\mathbb{I}_1}(\cdot b)}(a \cdot)$  and  $\text{coker}_{\ker_F(\cdot b)}(a \cdot) = \text{coker}_{\ker_{\mathbb{I}_1}(\cdot b)}(a \cdot)$ . Clearly, if  $\ker_{\mathbb{I}_1}(\cdot b) = 0$  then these vector spaces are equal to zero. Suppose that  $\ker_{\mathbb{I}_1}(\cdot b) \neq 0$ . Then  $\ker_{\mathbb{I}_1}(\cdot b) = \ker_F(\cdot b)$  since  $b \notin F$ , and  $\ker_{\mathbb{I}_1}(\cdot b) \simeq K[x]^m$  for some  $m \geq 1$ , by Theorem 3.6.(1a). Since  $a \in F$ , the kernel and the cokernel of the linear map  $a : K[x] \rightarrow K[x]$  are infinite dimensional vector spaces, then so are the vector spaces  $\ker_{\ker_{\mathbb{I}_1}(\cdot b)}(a \cdot)$  and  $\text{coker}_{\ker_{\mathbb{I}_1}(\cdot b)}(a \cdot)$ .

3(c). Let  $U := \ker_{\text{coker}_{\mathbb{I}_1}(\cdot b)}(a \cdot)$ . Suppose that  $\text{coker}_F(\cdot b) \neq 0$ . By Theorem 3.1.(1a),  ${}_{\mathbb{I}_1}\text{coker}_F(\cdot b) \simeq K[x]^m$  for some  $m \geq 1$  (since  $b \notin F$ ), and so  $\dim_K(\ker_{\text{coker}_F(\cdot b)}(a \cdot)) = \infty$  since  $a \in F$ . Since there is the inclusion  $\ker_{\text{coker}_F(\cdot b)}(a \cdot) \subseteq U$  (Lemma 4.1.(2b)), we must have  $\dim_K(U) = \infty$ .

Suppose that  $\text{coker}_F(\cdot b) = 0$ . Then the long exact sequence in Lemma 4.1.(2b) yields as isomorphism  $U \simeq \ker_{\text{coker}_{B_1}(\cdot b)}(a \cdot) = \text{coker}_{B_1}(\cdot b) \simeq B_1/B_1b$ . Notice that  $\text{coker}_F(\cdot b) = 0$  iff the map  $\cdot b : \mathbb{I}_1/\int \mathbb{I}_1 \rightarrow \mathbb{I}_1/\int \mathbb{I}_1$  is surjective since  $F_{\mathbb{I}_1} \simeq (\mathbb{I}_1/\int \mathbb{I}_1)^{(\mathbb{N})}$  iff the map  $(b^*)_{K[x]}$  is surjective.

$\dim_K(B_1/B_1b)$  is either 0 or  $\infty$ , and  $\dim_K(B_1/B_1b) = 0$  iff  $b = \lambda \int^i + f$ ,  $\lambda \partial^i + f$  where  $0 \neq \lambda \in K$ ,  $i \geq 0$  and  $f \in F$ . By Theorem 6.3, the two conditions,  $\text{coker}_F(\cdot b) = 0$  and  $\dim_K(B_1/B_1b) = 0$ , hold iff  $b = \lambda \int^i + f$  such that  $(b^*\cdot)_{K[x]}$  is a surjection where  $0 \neq \lambda \in K$ ,  $i \geq 0$  and  $f \in F$ . These are precisely the conditions when  $U = 0$ , otherwise  $\dim_K(U) = \infty$ .

3(d). Let  $V = \text{coker}_{\text{coker}_{\mathbb{I}_1}(\cdot b)}(a\cdot)$ . If  $\text{coker}_{B_1}(\cdot b) \neq 0$ , i.e.,  $\dim_K(B_1/B_1b) = \infty$ , then the end of the long exact sequence in Lemma 4.1.(2b) yields the surjection  $V \rightarrow \text{coker}_{\text{coker}_{B_1}(\cdot b)}(a\cdot) = \text{coker}_{B_1}(\cdot b) = B_1/B_1b$  (since  $a \in F$ ), and so  $\dim_K(V) = \infty$ .

If  $\text{coker}_{B_1}(\cdot b) = 0$  then the long exact sequence in Lemma 4.1.(2b) yields an isomorphism of vector spaces  $\text{coker}_{\text{coker}_F(\cdot b)}(a\cdot) \simeq V$ . Notice that the condition  $\text{coker}_{B_1}(\cdot b) = 0$  holds iff  $b = \lambda \int^i + f$ ,  $\lambda \partial^i + f$  where  $0 \neq \lambda \in K$ ,  $i \geq 0$  and  $f \in F$ .

If  $\text{coker}_F(\cdot b) = 0$  then  $V = 0$ . The condition  $\text{coker}_F(\cdot b) = 0$  holds iff the map  $\cdot b : \mathbb{I}_1 / \int \mathbb{I}_1 \rightarrow \mathbb{I}_1 / \int \mathbb{I}_1$  is surjective since  $F_{\mathbb{I}_1} \simeq (\mathbb{I}_1 / \int \mathbb{I}_1)^{(\mathbb{N})}$  iff the map  $(b^*\cdot)_{K[x]}$  is surjective. By Theorem 6.3, the two conditions,  $\text{coker}_{B_1}(\cdot b) = 0$  and  $\text{coker}_F(\cdot b) = 0$ , hold iff  $b = \lambda \int^i + f$  such that  $(b^*\cdot)_{K[x]}$  is a surjection where  $0 \neq \lambda \in K$ ,  $i \geq 0$  and  $f \in F$ .

If  $\text{coker}_F(\cdot b) \neq 0$  then, by Theorem 3.1.(1a),  $\mathbb{I}_1 \text{coker}_F(\cdot b) \simeq K[x]^m$  for some  $m \geq 1$  (since  $b \notin F$ ) and so  $\dim_K(V) = \dim_K(\text{coker}(a\cdot)_{K[x]^m}) = \infty$  since  $a \in F$ . This proves statement 3(d).

4. Since  $\mathbb{I}_1 F_{\mathbb{I}_1} \simeq K[x] \otimes (\mathbb{I}_1 / \int \mathbb{I}_1)$  and  $a, b \in F$ , the vector spaces at the beginning and the end of each left exact sequence in Lemma 4.1.(2a,b) are infinite dimensional, and so are the four vector spaces in statement 4.  $\square$

## 7 Classification of one-sided invertible elements of $\mathbb{I}_1$

In this section, a classification of elements of the algebra  $\mathbb{I}_1$  that admit a *one-sided* inverse is given (Corollary 7.2), an explicit description of all one-sided inverses is found (Theorem 7.5). It is proved that the monoid  $\mathcal{L}(\mathbb{I}_1)$  (respectively,  $\mathcal{R}(\mathbb{I}_1)$ ) of all elements of  $\mathbb{I}_1$  that admit a left (respectively, right) inverse is generated by the group  $\mathbb{I}_1^*$  of units of the algebra  $\mathbb{I}_1$  and the element  $\int$  (respectively,  $\partial$ ), Theorem 7.3.

The algebra  $K + F = K + M_\infty(K)$  has the obvious *determinant map*  $\det : K + F \rightarrow K$ ,  $a \mapsto \det(a)$ . The element  $a \in K + F$  is the unique finite sum  $a = \lambda + \sum \lambda_{ij} e_{ij} = \lambda + \sum \lambda_{ij} \frac{i!}{j!} E_{ij}$  where  $\lambda, \lambda_{ij} \in K$ . Then

$$\det(a) := \begin{cases} 0 & \text{if } \lambda = 0 \\ \lambda \det(a') & \text{if } \lambda \neq 0, \end{cases} = \begin{cases} 0 & \text{if } \lambda = 0 \\ \lambda \det(a'') & \text{if } \lambda \neq 0, \end{cases} \quad (33)$$

where  $a' := \sum_{i \in \mathbb{N}} e_{ii} + \sum \lambda^{-1} \lambda_{ij} e_{ij}$ ,  $a'' := \sum_{i \in \mathbb{N}} E_{ii} + \sum \lambda^{-1} \lambda_{ij} \frac{i!}{j!} E_{ij}$  (notice that  $a' = a'' = \lambda^{-1} a$ ). In the first equality the usual determinant is taken w.r.t. the matrix units  $\{e_{ij}\}_{i,j \in \mathbb{N}}$  but in the second equality - w.r.t. the matrix units  $\{E_{ij}\}_{i,j \in \mathbb{N}}$ . Both determinants are equal since for the element  $a \in K + F$ ,

$$[a]_e = S^{-1} [a]_E S,$$

where  $[a]_e$  and  $[a]_E$  are the matrix forms of the element  $a$  w.r.t. the matrix units  $\{e_{ij}\}$  and  $\{E_{ij}\}$  respectively, and  $S := \sum_{i \in \mathbb{N}} i! e_{ii} = \sum_{i \in \mathbb{N}} i! E_{ii}$  is the infinite diagonal invertible matrix.

**Theorem 7.1** 1. Let an element  $a \in \mathbb{I}_1$  be such that the map  $a_{K[x]}$  is an injection,  $n := \dim_K(\text{coker}(a_{K[x]}))$ , and  $b \in \mathbb{I}_1$ .

(a) Then  $ba \in \mathbb{I}_1^*$  iff  $a = a' \int^n$  where  $a' \in K^* + F$  and  $b = \partial^n b'$  where  $b' \in K^* + F$  with  $\det(ba) \neq 0$  (notice that  $ba \in K + F$ ).

(b) If elements  $a = a' \int^n$ , where  $n \in \mathbb{N}$ , and  $a' \in K^* + F$  are such that the map  $a_{K[x]}$  is injective (necessarily,  $n = \dim_K(\text{coker}(a_{K[x]}))$ , by Proposition 6.1.(1)) then there is at least one element  $b$  such that in the statement (a).

2. Let an element  $b \in \mathbb{I}_1$  be such that the map  $b_{K[x]}$  is a surjection,  $n := \dim_K(\ker(b_{K[x]}))$ , and  $a \in \mathbb{I}_1$ .

- (a) Then  $ba \in \mathbb{I}_1^*$  iff  $b = \partial^n b'$  where  $b' \in K^* + F$  and  $a = a' \int^n$  where  $a' \in K^* + F$  with  $\det(ba) \neq 0$ .
- (b) If elements  $b = \partial^n b'$ , where  $n \in \mathbb{N}$ , and  $b' \in K^* + F$  are such that the map  $b_{K[x]}$  is a surjection (necessarily,  $n = \dim_K(\ker(b_{K[x]}))$ ), by Proposition 6.1.(1)) then there is at least one element  $a$  such as in statement (a).

*Proof.* 1(a). ( $\Rightarrow$ ) Suppose that there exists an element  $b \in \mathbb{I}_1$  such that  $ba \in \mathbb{I}_1^*$ . Then necessarily the map  $b_{K[x]}$  is surjective. Then taking the inclusion  $ba \in \mathbb{I}_1^*$  modulo the ideal  $F$  yields the inclusion  $\bar{b}\bar{a} \in K^*$  where  $\bar{b} = b + F$  and  $\bar{a} = a + F$  since  $\mathbb{I}_1^* = K^*(1 + F)^*$ . Bearing in mind that  $a_{K[x]}$  is an injection and  $b_{K[x]}$  is a surjection, we must have  $a = a' \int^n$  where  $a' \in K^* + F$  (Theorem 6.6) and  $b = \partial^n b'$  where  $b' \in K^* + F$  (Corollary 6.4). Since  $ba \in \mathbb{I}_1^* = K^*(1 + F)^* = \{c \in K + F \mid \det(c) \neq 0\}$ , we must have  $\det(ba) \neq 0$ .

( $\Leftarrow$ ) Suppose that  $a$  and  $b$  satisfy the conditions after ‘iff’. Then  $\det(ba) \neq 0$  implies that  $ba \in \mathbb{I}_1^*$ .

1(b). By Theorem 6.11.(1), for the element  $a$  there exists an element  $b = \partial^n + f$  with  $f \in F$  such that the map  $(ba)_{K[x]}$  is a surjection and  $\ker(ba_{K[x]}) = \ker(a_{K[x]}) = 0$ , i.e.,  $ba \in \text{Aut}_K(K[x]) \cap (K + F) = (K + F)^* = \mathbb{I}_1^*$ .

2(a). ( $\Rightarrow$ ) Suppose that there is an element  $a \in \mathbb{I}_1$  such that  $ba \in \mathbb{I}_1^*$ . Then necessarily the map  $a_{K[x]}$  is injective. Then taking the inclusion modulo the ideal  $F$  yields the inclusion  $\bar{b}\bar{a} \in K^*$  where  $\bar{b} = b + F$  and  $\bar{a} = a + F$  since  $\mathbb{I}_1^* = K^*(1 + F)^*$ . Bearing in mind that  $a_{K[x]}$  is an injection and  $b_{K[x]}$  is a surjection, we must have  $a = a' \int^n$  where  $a' \in K^* + F$  (Theorem 6.6) and  $b = \partial^n b'$  where  $b' \in K^* + F$  (Corollary 6.4). Since  $ba \in \mathbb{I}_1^* = K^*(1 + F)^* = \{c \in K + F \mid \det(c) \neq 0\}$ , we must have  $\det(ba) \neq 0$ .

( $\Leftarrow$ ) Suppose that  $a$  and  $b$  satisfy the conditions after ‘iff’. Then  $\det(ba) \neq 0$  implies that  $ba \in \mathbb{I}_1^*$ .

2(b). By Theorem 6.11.(2), for the element  $b$  there is an element  $a = \int^n + g$  with  $g \in F$  such that the map  $(ba)_{K[x]}$  is an injection and  $\text{im}(ba_{K[x]}) = \text{im}(b_{K[x]}) = K[x]$ , i.e.,  $ba \in \text{Aut}_K(K[x]) \cap (K + F) = (K + F)^* = \mathbb{I}_1^*$ .  $\square$

Let  $\mathcal{L}(\mathbb{I}_1) := \{a \in \mathbb{I}_1 \mid ba = 1 \text{ for some } b \in \mathbb{I}_1\}$  and  $\mathcal{R}(\mathbb{I}_1) := \{b \in \mathbb{I}_1 \mid ba = 1 \text{ for some } a \in \mathbb{I}_1\}$ , i.e.,  $\mathcal{L}(\mathbb{I}_1)$  and  $\mathcal{R}(\mathbb{I}_1)$  are the sets of all the left and right invertible elements of the algebra  $\mathbb{I}_1$  respectively. The sets  $\mathcal{L}(\mathbb{I}_1)$  and  $\mathcal{R}(\mathbb{I}_1)$  are monoids and the group  $\mathbb{I}_1^*$  of invertible elements of the algebra  $\mathbb{I}_1$  is also the group of invertible elements of the monoids  $\mathcal{L}(\mathbb{I}_1)$  and  $\mathcal{R}(\mathbb{I}_1)$ ,  $\mathcal{L}(\mathbb{I}_1) \cap \mathcal{R}(\mathbb{I}_1) = \mathbb{I}_1^*$ . For an element  $u \in \mathbb{I}_1$ , let  $\text{l.inv}(u) := \{v \in \mathbb{I}_1 \mid vu = 1\}$  and  $\text{r.inv}(u) := \{v \in \mathbb{I}_1 \mid uv = 1\}$ , the *sets of left and right inverses* for the element  $u$ . The next theorem describes all the left and right inverses of elements in  $\mathbb{I}_1$ .

**Corollary 7.2** 1. An element  $a \in \mathbb{I}_1$  admits a left inverse iff  $a = a' \int^n$  for some natural number  $n \geq 0$  and an element  $a' \in K^* + F$  such that  $a_{K[x]}$  is an injection (necessarily,  $n = \dim_K(\text{coker}(a_{K[x]}))$ ). In this case,  $\text{l.inv}(a) = \{b = \partial^n b' \mid b' \in K^* + F, \int^n \partial^n b' a' \int^n \partial^n = \int^n \partial^n\}$ .  $\mathcal{L}(\mathbb{I}_1) = \{a \in (K^* + F) \int^n \mid n \in \mathbb{N}, a_{K[x]} \text{ is an injection}\}$ .

2. An element  $b \in \mathbb{I}_1$  admits a right inverse iff  $b \in \partial^n b'$  for some natural number  $n \geq 0$  and an element  $b' \in K^* + F$  such that  $b_{K[x]}$  is a surjection (necessarily,  $n = \dim_K(\ker(b_{K[x]}))$ ). In this case,  $\text{r.inv}(b) = \{a' \int^n \mid a' \in K^* + F, \int^n \partial^n b' a' \int^n \partial^n = \int^n \partial^n\}$ .  $\mathcal{R}(\mathbb{I}_1) = \{b \in \partial^n(K^* + F) \mid n \in \mathbb{N}, b_{K[x]} \text{ is a surjection}\}$ .

*Proof.* 1. Suppose that  $ba = 1$  for some element  $b \in \mathbb{I}_1$ . Then  $a_{K[x]}$  is an injection and  $b_{K[x]}$  is a surjection since  $\mathbb{I}_1 \subseteq \text{End}_K(K[x])$ . By Theorem 7.1.(1),  $a = a' \int^n$  and  $b = \partial^n b'$  for some elements  $a', b' \in K^* + F$  such that  $\partial^n b' a' \int^n = 1$ , or, equivalently,  $\int^n \partial^n b' a' \int^n \partial^n = \int^n \partial^n$ . The second equality is obtained from the first by applying  $\int^n (\cdot) \partial^n$ , and the first equality is obtained from the second by applying  $\partial^n (\cdot) \int^n$ . The last equality of statement 1 follows from Theorem 7.1.(1).

2. Suppose that  $ba = 1$  for some element  $a \in \mathbb{I}_1$ . Then  $a_{K[x]}$  is an injection and  $b_{K[x]}$  is a surjection. By Theorem 7.1.(2),  $a = a' \int^n$  and  $b = \partial^n b'$  for some elements  $a', b' \in K^* + F$  such

that  $\partial^n b' a' \int^n = 1$ , or, equivalently,  $\int^n \partial^n b' a' \int^n \partial^n = \int^n \partial^n$ . The last equality of statement 2 follows from Theorem 7.1.(2).  $\square$

Since  $\mathbb{I}_1 \subseteq \text{End}_K(K[x])$  we identify each element of the algebra  $\mathbb{I}_1$  with its matrix with respect to the basis  $\{x^i/i! \mid i \in \mathbb{N}\}$  of the vector space  $K[x]$ . The equality  $\int^n \partial^n b' a' \int^n \partial^n = \int^n \partial^n$  holds iff  $b' a' = \begin{pmatrix} A & B \\ C & 1_n \end{pmatrix}$  for some matrices  $A \in M_n(K)$ ,  $B$ ,  $C$  and  $1_n := e_{n,n} + e_{n+1,n+1} + \dots = 1 - e_{00} - \dots - e_{n-1,n-1}$ .

*Proof.*  $(\Rightarrow)$  Trivial.

$(\Leftarrow)$  For an arbitrary choice of the matrices  $A$ ,  $B$  and  $C$ ,

$$\partial^n \begin{pmatrix} A & B \\ C & 1_n \end{pmatrix} \int^n = \partial^n 1_n \int^n = \partial^n (1 - e_{00} - \dots - e_{n-1,n-1}) \int^n = 1. \quad \square$$

The set  $\mathcal{L}(\mathbb{I}_1)$  is the disjoint union

$$\mathcal{L}(\mathbb{I}_1) = \coprod_{n \in \mathbb{N}} \mathcal{L}(\mathbb{I}_1)_n \quad (34)$$

where  $\mathcal{L}(\mathbb{I}_1)_n := \{a \in (K^* + F) \mid a_{K[x]} \text{ is injective}\} = \{a' \int^n \mid a' \in K^* + F, \ker_{K[x]}(a') \cap (x^n) = 0\} = \{a \in \mathcal{L}(\mathbb{I}_1) \mid \dim_K(\text{coker}(a_{K[x]})) = n\} = \{a \in \mathcal{L}(\mathbb{I}_1) \mid -\text{ind}_{K[x]}(a) = n\}$  and  $\mathcal{L}(\mathbb{I}_1)_0 = \mathbb{I}_1^*$ , by Theorem 6.2.(3). Similarly, the set  $\mathcal{R}(\mathbb{I}_1)$  is the disjoint union

$$\mathcal{R}(\mathbb{I}_1) = \coprod_{n \in \mathbb{N}} \mathcal{R}(\mathbb{I}_1)_n \quad (35)$$

where  $\mathcal{R}(\mathbb{I}_1)_n := \{b \in \partial^n(K^* + F) \mid b_{K[x]} \text{ is surjective}\} = \{\partial^n b' \mid b' \in K^* + F, \text{im}_{K[x]}(b') + \sum_{i=0}^{n-1} Kx^i = K[x]\} = \{b \in \mathcal{R}(\mathbb{I}_1) \mid \dim_K(\ker(b_{K[x]})) = n\} = \{b \in \mathcal{R}(\mathbb{I}_1) \mid \text{ind}_{K[x]}(b) = n\}$  and  $\mathcal{R}(\mathbb{I}_1)_0 = \mathbb{I}_1^*$ , by Theorem 6.2.(3). Using the additivity of the index map  $\text{ind}_{K[x]}$  we see that  $\mathcal{L}(\mathbb{I}_1)$  and  $\mathcal{R}(\mathbb{I}_1)$  are  $\mathbb{N}$ -graded monoids, that is  $\mathcal{L}(\mathbb{I}_1)_n \mathcal{L}(\mathbb{I}_1)_m \subseteq \mathcal{L}(\mathbb{I}_1)_{n+m}$  and  $\mathcal{R}(\mathbb{I}_1)_n \mathcal{R}(\mathbb{I}_1)_m \subseteq \mathcal{R}(\mathbb{I}_1)_{n+m}$  for all  $n, m \in \mathbb{N}$ . In particular,  $\mathbb{I}_1^* \mathcal{L}(\mathbb{I}_1)_n \mathbb{I}_1^* = \mathcal{L}(\mathbb{I}_1)_n$  and  $\mathbb{I}_1^* \mathcal{R}(\mathbb{I}_1)_n \mathbb{I}_1^* = \mathcal{R}(\mathbb{I}_1)_n$  for all  $n \in \mathbb{N}$ . Since  $ab = 1$  iff  $b^* a^* = 1$ , we have the equalities

$$\mathcal{L}(\mathbb{I}_1)^* = \mathcal{R}(\mathbb{I}_1), \quad \mathcal{R}(\mathbb{I}_1)^* = \mathcal{L}(\mathbb{I}_1), \quad \mathcal{L}(\mathbb{I}_1)_n^* = \mathcal{R}(\mathbb{I}_1)_n, \quad \mathcal{R}(\mathbb{I}_1)_n^* = \mathcal{L}(\mathbb{I}_1)_n, \quad (36)$$

for all  $n \in \mathbb{N}$ . In particular, for all elements  $a \in \mathcal{L}(\mathbb{I}_1) \cup \mathcal{R}(\mathbb{I}_1)$ ,

$$\text{ind}_{K[x]}(a^*) = -\text{ind}_{K[x]}(a). \quad (37)$$

In general, the equality (37) does not hold.

*Example.*  $\text{ind}_{K[x]}(1 + \partial) = 0$  but  $\text{ind}_{K[x]}((1 + \partial)^*) = \text{ind}_{K[x]}(1 + \int) = -1$ .

Clearly,

$$\mathcal{L}(\mathbb{I}_1)_n \cap \mathcal{R}(\mathbb{I}_1)_m = \begin{cases} \mathbb{I}_1^* & \text{if } n = m = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

**Theorem 7.3** 1. The monoid  $\mathcal{L}(\mathbb{I}_1)$  is generated by the group  $\mathbb{I}_1^*$  and the element  $\int$ .

2. For all  $n \in \mathbb{N}$ ,  $\mathcal{L}(\mathbb{I}_1)_n = \mathbb{I}_1^* \int^n$ . For all  $n, m \in \mathbb{N}$ ,  $\mathcal{L}(\mathbb{I}_1)_n \mathcal{L}(\mathbb{I}_1)_m = \mathcal{L}(\mathbb{I}_1)_{n+m}$ .

3. The monoid  $\mathcal{R}(\mathbb{I}_1)$  is generated by the group  $\mathbb{I}_1^*$  and the element  $\partial$ .

4. For all  $n \in \mathbb{N}$ ,  $\mathcal{R}(\mathbb{I}_1)_n = \partial^n \mathbb{I}_1^*$ . For all  $n, m \in \mathbb{N}$ ,  $\mathcal{R}(\mathbb{I}_1)_n \mathcal{R}(\mathbb{I}_1)_m = \mathcal{R}(\mathbb{I}_1)_{n+m}$ .

*Proof.* 1. Statement 1 follows from statement 2 and (34).

2. It is obvious that  $\mathcal{L}(\mathbb{I}_1) \supseteq \mathbb{I}_1^* \int^n$ . Let  $a \in \mathcal{L}(\mathbb{I}_1)_n$ . To prove that the opposite inclusion holds we have to show that  $a = u \int^n$  for some  $u \in \mathbb{I}_1^*$ . Clearly,  $a = a' \int^n$  where  $a' \in K^* + F$  and  $\ker_{K[x]}(a') \cap (x^n) = 0$ . The last condition means that the columns  $c_n, c_{n+1}, \dots$  of the  $\mathbb{N} \times \mathbb{N}$  matrix  $a' \in K^* + F = K^* + M_\infty(K)$  are linearly independent (where  $c_i, i \in \mathbb{N}$  are the columns of the matrix  $a'$ ), and so there exists an element  $f = \sum_{j=0}^{n-1} \sum_{i \in \mathbb{N}} \lambda_{ij} e_{ij} \in F$ ,  $\lambda_{ij} \in K$ , such that all the columns of the matrix  $a' + f \in K^* + F = K^* + M_\infty(K)$  are linearly independent, i.e.,  $(a' + f) \in \mathbb{I}_1^*$ . The equality  $f \int^n = 0$  implies that  $(a' + f) \int^n = a' \int^n = a$ , i.e.,  $\mathcal{L}(\mathbb{I}_1)_n = \mathbb{I}_1^* \int^n$ . Then  $\mathcal{L}(\mathbb{I}_1)_n \mathcal{L}(\mathbb{I}_1)_m = \mathbb{I}_1^* \int^n \mathbb{I}_1^* \int^m = \mathbb{I}_1^* \int^{n+m} = \mathcal{L}(\mathbb{I}_1)_{n+m}$  for all  $n, m \in \mathbb{N}$ .

3 and 4. Statements 3 and 4 are obtained from statements 1 and 2 by applying the involution  $*$  of the algebra  $\mathbb{I}_1$ , see (36).  $\square$

*Remark.* For all  $n \geq 1$ ,  $\mathcal{L}(\mathbb{I}_1)_n = \mathbb{I}_1^* \int^n \supsetneq \int^n \mathbb{I}_1^*$  and  $\mathcal{R}(\mathbb{I}_1)_n = \partial^n \mathbb{I}_1^* \supsetneq \mathbb{I}_1^* \partial^n$ , Corollary 7.6.

**Corollary 7.4** 1. The decomposition  $\mathcal{L}(\mathbb{I}_1) = \bigsqcup_{n \in \mathbb{N}} \mathcal{L}(\mathbb{I}_1)_n$  is the orbit decomposition of the action of the group  $\mathbb{I}_1^*$  on  $\mathcal{L}(\mathbb{I}_1)$  by left multiplication,  $\mathcal{L}(\mathbb{I}_1)_n = \mathbb{I}_1^* \int^n$  and the stabilizer of  $\int^n$  is equal to  $\text{st}_{\mathbb{I}_1^*}(\int^n) = \begin{pmatrix} \text{GL}_n(K) & 0 \\ * & 1 \end{pmatrix} \subseteq (1 + F)^*$ . In particular, there are only countably many orbits and the action of  $\mathbb{I}_1^*$  is not free.

2. The decomposition  $\mathcal{R}(\mathbb{I}_1) = \bigsqcup_{n \in \mathbb{N}} \mathcal{R}(\mathbb{I}_1)_n$  is the orbit decomposition of the action of the group  $\mathbb{I}_1^*$  on  $\mathcal{R}(\mathbb{I}_1)$  by right multiplication,  $\mathcal{R}(\mathbb{I}_1)_n = \partial^n \mathbb{I}_1^*$  and the stabilizer of  $\partial^n$  is equal to  $\text{st}_{\mathbb{I}_1^*}(\partial^n) = \begin{pmatrix} \text{GL}_n(K) & * \\ 0 & 1 \end{pmatrix} \subseteq (1 + F)^*$ . In particular, there are only countably many orbits and the action of  $\mathbb{I}_1^*$  is not free.

The next theorem describes in explicit terms all the left and right inverses for elements of  $\mathbb{I}_1$ .

**Theorem 7.5** 1. Let  $a \in \mathcal{L}(\mathbb{I}_1)$ , i.e.,  $a = a' \int^n$  where  $a' \in \mathbb{I}_1^*$  and  $n \in \mathbb{N}$  (Theorem 7.3.(2)). Then  $\text{l.inv}(a) = (\partial^n + E_{\mathbb{N},0} + \dots + E_{\mathbb{N},n-1})a'^{-1}$  where  $E_{\mathbb{N},i} := \sum_{j \in \mathbb{N}} K e_{ji} = \sum_{j \in \mathbb{N}} K E_{ji}$ .

2. Let  $b \in \mathcal{R}(\mathbb{I}_1)$ , i.e.,  $b = \partial^n b'$  where  $b' \in \mathbb{I}_1^*$  and  $n \in \mathbb{N}$  (Theorem 7.3.(4)). Then  $\text{r.inv}(a) = b'^{-1}(\int^n + E_{0,\mathbb{N}} + \dots + E_{n-1,\mathbb{N}})$  where  $E_{i,\mathbb{N}} := \sum_{j \in \mathbb{N}} K e_{ij} = \sum_{j \in \mathbb{N}} K E_{ij}$ .

*Proof.* 1. It is obvious that  $\text{l.inv}(a) = \partial^n a'^{-1} + \ker_{\mathbb{I}_1}(\cdot \int^n)$ . Since  $\ker_{\mathbb{I}_1}(\cdot \int^n) = E_{\mathbb{N},0} + \dots + E_{\mathbb{N},n-1} =: \mathcal{E}$  and  $a' \in \mathbb{I}_1^*$ ,  $\ker_{\mathbb{I}_1}(\cdot a' \int^n) = \mathcal{E} a'^{-1}$ . Therefore,  $\text{l.inv}(a) = (\partial^n + \mathcal{E})a'^{-1}$ .

2. Statement 2 follows from statement 1 by applying the involution  $*$  and using the equalities  $\mathcal{L}(\mathbb{I}_1)_n^* = \mathcal{R}(\mathbb{I}_1)_n$  and  $\text{r.inv}(a^*) = \text{l.inv}(a)^*$ .  $\square$

Recall that  $\mathbb{I}_1^* = K^*(1 + F)^* = K^* \times (1 + F)^* \simeq K^* \times \text{GL}_\infty(K)$ . We use the matrix units  $\{e_{ij}\}_{i,j \in \mathbb{N}}$  to define the isomorphism  $(e_{ij} \leftrightarrow E_{ij})$ . Define the group monomorphism  $\kappa : \mathbb{I}_1^* \rightarrow \mathbb{I}_1^*$  by the rule: for all  $\lambda \in K^*$  and  $u \in (1 + F)^* \simeq \text{GL}_\infty(K)$ ,

$$\kappa(\lambda) = \lambda, \quad \kappa(u) = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}. \quad (38)$$

By the very definition,  $\kappa(K^*) = K^*$  and  $\kappa((1 + F)^*) \subseteq (1 + F)^*$ . Moreover,  $\kappa(u) = e_{00} + \int u \partial$  and, by induction,

$$\kappa^n(u) = e_{00} + e_{11} + \dots + e_{n-1,n-1} + \int^n u \partial^n = \begin{pmatrix} E_n & 0 \\ 0 & u \end{pmatrix} \quad (39)$$

where  $E_n = e_{00} + e_{11} + \dots + e_{n-1,n-1}$  is the  $n \times n$  identity matrix. Clearly,

$$\kappa(u^*) = \kappa(u)^* \quad (40)$$

and

$$\det(\kappa(u)) = \det(u) \quad (41)$$

where the determinant map  $\det : (1 + F)^* \rightarrow (1 + F)^*$  is taken with respect to the basis  $\{e_{ij}\}$  (or  $\{E_{ij}\}$ ; both determinants coincide, see (33)). For all  $u \in (1 + F)^*$  and all  $n \geq 1$ ,

$$\int^n u = \kappa^n(u) \int^n \quad \text{and} \quad u \partial^n = \partial^n \kappa^n(u). \quad (42)$$

By (39), (40) and  $(\mathbb{I}_1^*)^* = \mathbb{I}_1^*$ , it suffices to prove that  $\int u = \kappa(u) \int$ :  $\int u = \int u \partial \int = (e_{00} + \int u \partial) \int = \kappa(u) \int$ . Let  $u, v \in (1 + F)^*$ . Then

$$\kappa^n(u) \int^n = \kappa(v) \int^n \Rightarrow u = v. \quad (43)$$

*Proof.* The equality can be written as  $\int^n u = \int^n v$ . Then multiplying this equality by the element  $\partial^n$  on the left yields the result.  $\square$

**Corollary 7.6** 1. For all  $n \geq 1$ ,  $\mathcal{L}(\mathbb{I}_1)_n = \mathbb{I}_1^* \int^n \not\subseteq \int^n \mathbb{I}_1^*$ . The right action of the group  $\mathbb{I}_1^*$  on the set  $\mathcal{L}(\mathbb{I}_1)$  is free (i.e., the stabilizer of each point is the identity group).

2. For all  $n \geq 1$ ,  $\mathcal{R}(\mathbb{I}_1)_n = \partial^n \mathbb{I}_1^* \not\subseteq \mathbb{I}_1^* \partial^n$ . The left action of the group  $\mathbb{I}_1^*$  on the set  $\mathcal{R}(\mathbb{I}_1)$  is free.

*Proof.* 1. By (39) and (42),  $\mathcal{L}(\mathbb{I}_1)_n = \mathbb{I}_1^* \int^n \not\subseteq \int^n \mathbb{I}_1^*$ . The freeness of the right action of the group  $\mathbb{I}_1^*$  follows from (34) and the facts that  $\mathcal{L}(\mathbb{I}_1)_n = \mathbb{I}_1^* \int^n$ ,  $\mathcal{L}(\mathbb{I}_1)_n \mathbb{I}_1^* \subseteq \mathcal{L}(\mathbb{I}_1)_n$ : if  $u \int^n v = u \int^n$  where  $u, v \in \mathbb{I}_1^*$  then  $\int^n v = \int^n$ , and so  $v = 1$ .

2. Applying the involution  $*$  to statement 1 yields statement 2.  $\square$

## 8 The algebras $\widetilde{\mathbb{I}}_1$ and $\widetilde{\mathbb{J}}_1$

The aim of this section is to introduce and study the algebras  $\widetilde{\mathbb{I}}_1$  and  $\widetilde{\mathbb{J}}_1$  that have remarkable properties (Theorem 8.3 and Corollary 8.5). Name just a few: both algebras are obtained from  $\mathbb{I}_1$  by inverting certain elements; they contain the only proper ideal,  $\mathcal{C}(\widetilde{\mathbb{I}}_1) \triangleleft \widetilde{\mathbb{I}}_1$  and  $\mathcal{C}(\widetilde{\mathbb{J}}_1) \triangleleft \widetilde{\mathbb{J}}_1$ ; the factor algebras  $\widetilde{\mathbb{I}}_1/\mathcal{C}(\widetilde{\mathbb{I}}_1)$  and  $\widetilde{\mathbb{J}}_1/\mathcal{C}(\widetilde{\mathbb{J}}_1)$  are canonically isomorphic to the skew field of fractions  $\text{Frac}(A_1)$  of the Weyl algebra  $A_1$  and its opposite skew field  $\text{Frac}(A_1)^{op}$  respectively. The algebras  $\widetilde{\mathbb{I}}_1$  and  $\widetilde{\mathbb{J}}_1$  will turn out to be the largest right and the largest left quotients rings of the algebra  $\mathbb{I}_1$  respectively (Theorem 9.7).

Let  $V$  be an infinite dimensional vector space. The set  $\mathcal{V} = \mathcal{V}(V)$  of all vector subspaces  $U$  of  $V$  of finite codimension, i.e.,  $\text{codim}(U) := \dim_K(V/U) < \infty$ , is a filter. This means that

- (i) if  $U \subseteq W$  are subspaces of  $V$  and  $U \in \mathcal{V}$  then  $W \in \mathcal{V}$ ;
- (ii) if  $U_1, \dots, U_n \in \mathcal{V}$  then  $\bigcap_{i=1}^n U_i \in \mathcal{V}$ ;
- (iii)  $\emptyset \notin \mathcal{V}$ .

Let  $\mathcal{F}(V)$  be the set of all Fredholm linear maps/operators in an infinite dimensional space in  $V$ . The set  $\mathcal{F}(V)$  is a monoid with respect to composition of maps,  $\mathcal{F}(V) + \mathcal{C}(V) = \mathcal{F}(V)$  and  $\mathcal{F}(V) \cap \mathcal{C}(V) = \emptyset$ . For each element  $f \in \mathcal{F}(V)$  and for each vector space  $U \in \mathcal{V}(V)$ ,  $f(U), f^{-1}(U) \in \mathcal{V}(V)$ . Let  $A$  be a subalgebra of  $\text{End}_K(V)$  and  $\mathcal{C}(A) := A \cap \mathcal{C}(V)$  be the ideal of compact operators of the algebra  $A$ . Suppose that  $A \setminus \mathcal{C}(A) \subseteq \mathcal{F}(V)$ . Example:  $V = K[x]$ ,  $A = \mathbb{I}_1 \subseteq \text{End}_K(K[x])$  and  $\mathcal{C}(\mathbb{I}_1) = F$  (Theorem 3.1, Corollary 3.3). Note that the disjoint union  $\mathcal{F}(V) \sqcup \mathcal{C}(V)$  is a semigroup but *not* a ring but  $\mathcal{F}(\mathbb{I}_1) \sqcup \mathcal{C}(\mathbb{I}_1) = \mathbb{I}_1$  is a ring. A subalgebra  $A$  of  $\text{End}_K(V)$  satisfies the condition  $A \setminus \mathcal{C}(A) \subseteq \mathcal{F}(V)$  iff  $A \subseteq \mathcal{F}(V) \sqcup \mathcal{C}(V)$  iff the Compact-Fredholm Alternative holds for the left  $A$ -module  $V$ . For such a subalgebra  $A$ , we say that two elements  $a$  and  $b$  of  $A$  are *equivalent*,  $a \sim b$ , if  $a|_V = b|_V$  for some  $V \in \mathcal{V}(V)$ . The relation  $\sim$  is an equivalent relation on the algebra  $A$ . Let  $[a]$  be the equivalence class of the element  $a$ . Then the

set of equivalence classes  $\overline{A} = \{[a] \mid a \in A\}$  is a  $K$ -algebra where  $[a] + [b] := [a + b]$ ,  $[a][b] := [ab]$  and  $\lambda[a] = [\lambda a]$  for all  $a, b \in A$  and  $\lambda \in K$  (the inclusion  $A \setminus \mathcal{C}(A) \subseteq \mathcal{F}(A)$  guarantees that the multiplication is well defined). Since  $[a] = 0$  iff  $a \in \mathcal{C}(A)$ , there is a canonical algebra isomorphism  $A/\mathcal{C}(A) \rightarrow \overline{A}$ ,  $a + \mathcal{C}(A) \mapsto [a]$ . The inclusion  $A \setminus \mathcal{C}(A) \subseteq \mathcal{F}(A)$  implies that the algebra  $\overline{A}$  is a domain. Similarly, the set  $\overline{\mathcal{F}}(V) := \mathcal{F}(V)/\mathcal{C}(V) = \{[a] = a + \mathcal{C}(V) \mid a \in \mathcal{F}(V)\}$  is a monoid,  $[a][b] = [ab]$ . For vector spaces  $U$  and  $W$ ,  $\text{Iso}_K(U, W)$  is the set of all bijective linear maps from  $U$  to  $W$ . Via the map  $[a] \mapsto [a]$ , the monoid  $\overline{\mathcal{F}}(V)$  is canonically isomorphic to the group  $\mathcal{G}(V) = \{[a] \mid a \in \text{Iso}_K(U, W) \text{ for some } U, W \in \mathcal{V}(V)\}$ ;  $[a] = [a']$  where  $a' \in \text{Hom}_K(U', W')$  and  $U', W' \in \mathcal{V}(V)$  iff  $a|_{U''} = a'|_{U''}$  for some  $U'' \in \mathcal{V}(V)$  such that  $U'' \subseteq U \cap U'$ ; and for  $b \in \text{Iso}_K(U', W')$  and  $b \in \mathcal{G}(V)$ ,  $[ab] = [ab|_{b^{-1}(W' \cap U)}]$  and  $[a]^{-1} = [a^{-1} : W \rightarrow U]$ . It is convenient to identify the groups  $\mathcal{F}(V)$  and  $\mathcal{G}(V)$  via the group isomorphism  $[a] \mapsto [a]$ .

*Definition.* Let the algebra  $A$  be as above, i.e.,  $A \subseteq \text{End}_K(V)$  and  $A \setminus \mathcal{C}(A) \subseteq \mathcal{F}(V)$ . Then  $\text{frac}_V(A) := \text{frac}_V(\overline{A})$  is the intersection of all subalgebras  $B$  (if they exist) of the set  $\mathcal{G}(V) \cup \{0\}$  such that  $\overline{A}, \overline{A}^{-1} \subseteq B$  where  $\overline{A}^{-1} := \{[a]^{-1} \mid 0 \neq [a] \in \overline{A}\}$ . So,  $\text{frac}_V(\overline{A})$  is the subalgebra (if it exists) in  $\mathcal{G}(V) \cup \{0\}$  generated by  $\overline{A}$  and  $\overline{A}^{-1}$ .

**Proposition 8.1** *Let the algebra  $A$  be as above, i.e.,  $A \subseteq \text{End}_K(V)$  and  $A \setminus \mathcal{C}(A) \subseteq \mathcal{F}(A)$ . Suppose that  $\overline{A}$  is a left (resp. right) Goldie domain and  $\text{Frac}(\overline{A})$  be its left (resp. right) skew field of fractions. Then the map  $\text{Frac}(\overline{A}) \rightarrow \text{frac}_V(\overline{A})$ ,  $[s]^{-1}[a] \mapsto [s]^{-1}[a]$  (resp.  $[a][s]^{-1} \mapsto [a][s]^{-1}$ ) is an algebra isomorphism.*

*Proof.* The statement follows from the left (resp. right) Ore condition and the fact that each nonzero element of the algebra  $\overline{A}$  is invertible in  $\mathcal{G}(V)$ . In more detail, let  $\overline{A}$  be a left (resp. right) Goldie domain. The left (resp. right) Ore condition in the domain  $\overline{A}$  implies that each element of  $\text{Frac}(\overline{A})$  and of  $\text{frac}_V(\overline{A})$  has the form  $[s]^{-1}[a]$  (resp.  $[a][s]^{-1}$ ) where  $[a] \in \overline{A}$  and  $0 \neq [s] \in \overline{A}$ . By the universal property of the quotient ring  $\text{Frac}(\overline{A})$ , the map  $[s]^{-1}[a] \mapsto [s]^{-1}[a]$  (resp.  $[a][s]^{-1} \mapsto [a][s]^{-1}$ ) is an algebra epimorphism with zero kernel, i.e., an isomorphism.  $\square$

**Corollary 8.2** 1. For all  $\mathbb{I}_1$ -modules  $M$  of finite length,  $\text{frac}_M(\mathbb{I}_1) \simeq \text{Frac}(B_1) = \text{Frac}(A_1)$ .

2. For all  $A_1$ -modules  $M$  of finite length,  $\text{frac}_M(A_1) \simeq \text{Frac}(A_1)$ .

*Proof.* 1. This follows from Theorem 3.1 and Proposition 8.1.  
2. This follows from [25] and Proposition 8.1.  $\square$

Recall that the algebra  $B_1 = K[H][\partial, \partial^{-1}; \tau]$  is the (left and right) localization of the Weyl algebra  $A_1$  at the powers of the element  $\partial$ ,  $B_1 = \mathcal{S}_\partial^{-1}A_1$ , and  $\text{Frac}(B_1) = \text{Frac}(A_1)$ . Moreover, the multiplicatively closed set  $B_1^0 := \{\sum_{i \geq 0} a_{-i}\partial^i \in B_1 \mid a_0 \neq 0, \text{ all } a_j \in K[H]\}$  is a left and right Ore set in  $B_1$  such that

$$\text{Frac}(A_1) = \text{Frac}(B_1) = B_1^{0^{-1}}B_1 = B_1B_1^{0^{-1}}. \quad (44)$$

Notice that under the natural algebra epimorphism  $\pi : \mathbb{I}_1 \rightarrow B_1 = \mathbb{I}_1/F$ ,  $a \mapsto \overline{a} := a + F$ ,

$$\pi(\mathbb{I}_1^0) = B_1^0, \quad (45)$$

by Theorem 6.2.(1), where  $\mathbb{I}_1^0 := \mathbb{I}_1 \cap \text{Aut}_K(K[x])$ , the multiplicative submonoid of the group  $\text{Aut}_K(K[x])$ . Let  $\widetilde{\mathbb{I}}_1$  be the subalgebra of  $\text{End}_K(K[x])$  generated by  $\mathbb{I}_1$  and  $\mathbb{I}_1^0{}^{-1}$ , i.e., it is obtained from the algebra  $\mathbb{I}_1$  by adding the inverse elements of all the elements of the set  $\mathbb{I}_1^0$ .

**Theorem 8.3** 1.  $\mathbb{I}_1^{0^{-1}}F = F$  but  $F \subsetneq F\mathbb{I}_1^{0^{-1}}$ .

2. The multiplicatively closed set  $\mathbb{I}_1^0$  is a right but not left Ore subset of  $\mathbb{I}_1$  and  $\mathbb{I}_1\mathbb{I}_1^{0^{-1}} \simeq \widetilde{\mathbb{I}}_1$ .

3.  $F\mathbb{I}_1^{0^{-1}}$  is the only proper ideal of the algebra  $\widetilde{\mathbb{I}}_1$ ,  $(F\mathbb{I}_1^{0^{-1}})^2 = F\mathbb{I}_1^{0^{-1}}$ .

4.  $\tilde{\mathbb{I}}_1/F\mathbb{I}_1^{0-1} \simeq \text{Frac}(A_1)$ .
5.  $\mathcal{C}(\tilde{\mathbb{I}}_1) = F\mathbb{I}_1^{0-1}$  (where  $\mathcal{C}(\tilde{\mathbb{I}}_1) := \tilde{\mathbb{I}}_1 \cap \mathcal{C}(K[x])$ ).
6.  $\text{frac}_{K[x]}(\tilde{\mathbb{I}}_1) \simeq \tilde{\mathbb{I}}_1/\mathcal{C}(\tilde{\mathbb{I}}_1) \simeq \text{Frac}(A_1)$ .
7. (a) There are only two (up to isomorphism) simple left  $\tilde{\mathbb{I}}_1$ -modules,  $K[x]$  and  $\text{Frac}(A_1) = \tilde{\mathbb{I}}_1/\mathcal{C}(\tilde{\mathbb{I}}_1)$ . The first one is faithful and the second one is not.  
(b) There are only two (up to isomorphism) simple right  $\tilde{\mathbb{I}}_1$ -modules,  $(\mathbb{I}_1/\int \mathbb{I}_1)\mathbb{I}_1^{0-1} \simeq K[\partial]\mathbb{I}_1^{0-1}$  and  $\text{Frac}(A_1)$ . The first one is faithful and the second one is not.
8.  ${}_{\tilde{\mathbb{I}}_1}F = \bigoplus_{i \in \mathbb{N}} E_{\mathbb{N},i}$  is an infinite direct sum of nonzero left ideals  $E_{\mathbb{N},i} = Fe_{0i} \simeq K[x]$ , therefore the algebra  $\tilde{\mathbb{I}}_1$  is not left Noetherian.  $F\mathbb{I}_1^{0-1} = (\bigoplus_{i \in \mathbb{N}} E_{i,\mathbb{N}})\mathbb{I}_1^{0-1} \simeq \bigoplus_{i \in \mathbb{N}} E_{i,\mathbb{N}}\mathbb{I}_1^{0-1}$  is an infinite direct sum of nonzero right ideals  $E_{i,\mathbb{N}}\mathbb{I}_1^{0-1}$  where  $E_{i,\mathbb{N}} = e_{i0}F$ , therefore the algebra  $\tilde{\mathbb{I}}_1$  is not right Noetherian.

*Proof.* 1. The equality  $\mathbb{I}_1^{0-1}F = F$  follows from the inversion formula (Theorem 6.2.(4)):  
 $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} a' & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} aa' & 0 \\ 0 & 0 \end{pmatrix} \in F$ . The inclusion  $F \subset F\mathbb{I}_1^{0-1}$  is strict as  $e_{00}(1 - \partial)^{-1} = e_{00} \sum_{i \geq 0} \partial^i = \sum_{i \geq 0} e_{0i} \in F\mathbb{I}_1^{0-1} \setminus F$ .

2. Notice that all elements of the set  $\mathbb{I}_1^0$  are (left and right) regular, i.e., non-zero divisors in  $\mathbb{I}_1$ . To prove that  $\mathbb{I}_1^0$  is a right Ore set we have to show that for all elements  $a \in \mathbb{I}_1$  and  $s \in \mathbb{I}_1^0$  there exist elements  $b \in \mathbb{I}_1$  and  $t \in \mathbb{I}_1^0$  such that  $at = sb$ . By (44) and (45),  $at = sb_1 + f$  for some elements  $t \in \mathbb{I}_1^0$ ,  $b_1 \in \mathbb{I}_1$  and  $f \in F$ . By statement 1,  $s_1^{-1}f \in F$ , and so  $at = s(b_1 + s^{-1}f) = sb$  where  $b = b_1 + s^{-1}f \in \mathbb{I}_1$ , as required. Recall that the algebra  $\tilde{\mathbb{I}}_1$  is the subalgebra of the endomorphism algebra  $\text{End}_K(K[x])$  generated by the algebra  $\mathbb{I}_1$  and  $\mathbb{I}_1^{0-1}$ . Since  $\mathbb{I}_1^0$  is a right Ore set, there is a natural algebra epimorphism  $\mathbb{I}_1\mathbb{I}_1^{0-1} \rightarrow \tilde{\mathbb{I}}_1$  such that its restriction to the subalgebra  $\mathbb{I}_1$  gives the identity map  $\mathbb{I}_1 \rightarrow \mathbb{I}_1 \subseteq \tilde{\mathbb{I}}_1$ . Since the subalgebra  $\mathbb{I}_1$  is an essential right  $\mathbb{I}_1$ -module of  $\mathbb{I}_1\mathbb{I}_1^{0-1}$ , the epimorphism is necessarily an isomorphism.

The set  $\mathbb{I}_0$  is not a left Ore set, otherwise we would have the equality  $\mathbb{I}_1^{0-1}F = F\mathbb{I}_1^{0-1}$  which would have contradicted to statement 1 (the equality follows easily from the fact that  $F$  is an ideal of  $\mathbb{I}_1$  and the algebra  $\mathbb{I}_1/F$  is a domain. In more detail, we have to show that every fraction  $s^{-1}f$  (resp.  $fs^{-1}$ ) where  $s \in \mathbb{I}_1^0$  and  $f \in F$  can be written as  $gt^{-1}$  (resp.  $t^{-1}g$ ) for some  $t \in \mathbb{I}_1^0$  and  $g \in F$ . Notice that  $s^{-1}f = gt^{-1}$  (resp.  $fs^{-1} = t^{-1}g$ ) for some  $t \in \mathbb{I}_1^0$  and  $g \in \mathbb{I}_1$ , i.e.,  $ft = sg \in F$  (resp.  $tf = gs \in F$ ). Since  $\mathbb{I}_1/F$  is a domain and  $s \in \mathbb{I}_1^0$  we conclude that  $g \in F$  (resp.  $g \in F$ ).

3. By statement 1,  $F\mathbb{I}_1^{0-1} = \mathbb{I}_1\mathbb{I}_1^{0-1}F\mathbb{I}_1^{0-1}$  is a proper ideal of the algebra  $\mathbb{I}_1\mathbb{I}_1^{0-1}$  (since  $0 \neq F \subseteq F\mathbb{I}_1^{0-1}$  and  $F\mathbb{I}_1^{0-1} \neq \mathbb{I}_1\mathbb{I}_1^{0-1}$  as  $F \cap \mathbb{I}_1^0 = \emptyset$ ). Therefore, by statement 1,  $(F\mathbb{I}_1^{0-1})^2 = F\mathbb{I}_1^{0-1}F\mathbb{I}_1^{0-1} = F^2\mathbb{I}_1^{0-1} = F\mathbb{I}_1^{0-1}$  since  $F = F^2$ . Since  $F$  is the only proper ideal of the algebra  $\mathbb{I}_1$  and an essential  $\mathbb{I}_1$ -bimodule [12], the ideal  $F\mathbb{I}_1^{0-1}$  is the only proper ideal of the algebra  $\tilde{\mathbb{I}}_1$ .

4.  $\tilde{\mathbb{I}}_1/F\mathbb{I}_1^{0-1} = \mathbb{I}_1\mathbb{I}_1^{0-1}/F\mathbb{I}_1^{0-1} \simeq (\mathbb{I}_1/F)\mathbb{I}_1^{0-1} \stackrel{(45)}{=} B_1B_1^{0-1} = \text{Frac}(B_1) = \text{Frac}(A_1)$ .

5. Since  $F\mathbb{I}_1^{0-1} \subseteq \mathcal{C}(K[x])$ , we have the inclusion  $F\mathbb{I}_1^{0-1} \subseteq \tilde{\mathbb{I}}_1 \cap \mathcal{C}(K[x]) = \mathcal{C}(\tilde{\mathbb{I}}_1)$  of proper ideals of the algebra  $\tilde{\mathbb{I}}_1$ . By statement 3, the inclusion is the equality.

6. By statements 4 and 5, the factor algebra  $\tilde{\mathbb{I}}_1/\mathcal{C}(\tilde{\mathbb{I}}_1) = \tilde{\mathbb{I}}_1/F\mathbb{I}_1^{0-1} \simeq \text{Frac}(A_1)$  is a skew field. Therefore,  $\text{frac}_{K[x]}(\tilde{\mathbb{I}}_1) = \tilde{\mathbb{I}}_1/\mathcal{C}(\tilde{\mathbb{I}}_1) \simeq \text{Frac}(A_1)$ .

7(a). Let  $M$  be a simple left  $\tilde{\mathbb{I}}_1$ -module. Then either  $F\mathbb{I}_1^{0-1}M = 0$  or  $F\mathbb{I}_1^{0-1}M = M$ . In the first case,  $M \simeq \tilde{\mathbb{I}}_1/F\mathbb{I}_1^{0-1} \simeq \text{Frac}(A_1)$ . In the second case,  $FM \neq 0$ . Since  $F = \sum_{i \geq 0} Fe_{0i}$ , must have  $e_{0i}m \neq 0$  for some nonzero element  $m$  of  $M$  and some natural number  $i$ . Then  ${}_{\tilde{\mathbb{I}}_1}Fe_{0i}m \simeq K[x]$  since  $\mathbb{I}_1^{0-1}F = F$ , by statement 1.

7(b). Let  $N$  be a simple right  $\tilde{\mathbb{I}}_1$ -module. Then either  $NF\mathbb{I}_1^{0-1} = 0$  or  $NF\mathbb{I}_1^{0-1} = N$ . In the first case,  $N \simeq \tilde{\mathbb{I}}_1/F\mathbb{I}_1^{0-1} \simeq \text{Frac}(A_1)$ . In the second case,  $NF \neq 0$ . Since  $F = \sum_{i \geq 0} e_{i0}F$ ,



must have  $ne_{i0} \neq 0$  for some nonzero element  $n$  of  $N$  and some natural number  $i$ . Then the right  $\mathbb{I}_1$ -module  $ne_{i0}F \simeq \mathbb{I}_1 / \int \mathbb{I}_1 \simeq K[\partial]$  is simple, hence so is its localization  $ne_{i0}F\mathbb{I}_1^{0-1}$  provided it is not equal to zero which is the case as  $0 \neq n \in N_{\mathbb{I}_1}$ .

8. Statement 8 is obvious.  $\square$

Theorem 8.3 shows that the algebra  $\widetilde{\mathbb{I}}_1$  is not left-right symmetric. In particular, the involution  $*$  of the algebra  $\mathbb{I}_1$  *cannot* be extended to an involution of the algebra  $\widetilde{\mathbb{I}}_1$  since otherwise we would have the inclusion  $(\mathbb{I}_1^0)^* \subseteq \text{Aut}_K(K[x])$  which is not the case as  $1 + \partial \in \mathbb{I}_1^0$  but  $(1 + \partial)^* = 1 + \int \notin \text{Aut}_K(K[x])$ , by Theorem 6.2.(1).

In fact, we *can* ‘extend’ the involution  $*$  of the algebra  $\mathbb{I}_1$  in the following sense: we will construct an algebra  $\widetilde{\mathbb{J}}_1$ , that contains the algebra  $\mathbb{I}_1$ , and two  $K$ -algebra *anti-isomorphisms*

$$*: \widetilde{\mathbb{I}}_1 \rightarrow \widetilde{\mathbb{J}}_1, \quad a \mapsto a^*, \quad *: \widetilde{\mathbb{J}}_1 \rightarrow \widetilde{\mathbb{I}}_1, \quad b \mapsto b^*, \quad (46)$$

such that  $a^{**} = a$ ,  $b^{**} = b$ ,  $\mathbb{I}_1^* = \mathbb{I}_1$  and the restriction of the map  $*$  in (46) to the subalgebra  $\mathbb{I}_1$  coincides with the involution  $*$ . Recall that a  $K$ -algebra *anti-isomorphism*  $\varphi : A \rightarrow B$  is a bijective linear map such that  $\varphi(uv) = \varphi(v)\varphi(u)$  for all elements  $u, v \in A$ ; equivalently,  $\varphi : A \rightarrow B^{op}$  is a  $K$ -algebra isomorphism where  $B^{op}$  is the opposite algebra to  $B$ . The equalities  $(\mathbb{I}_1\partial)^* = \int \mathbb{I}_1$  and  $(\int \mathbb{I}_1)^* = \mathbb{I}_1\partial$  yield the  $K$ -linear isomorphisms:

$$K[\int] \simeq \mathbb{I}_1/\mathbb{I}_1\partial \xrightarrow{*} \mathbb{I}_1/\int \mathbb{I}_1 \simeq K[\partial], \quad p + \mathbb{I}_1\partial \mapsto p^* + \int \mathbb{I}_1, \quad (47)$$

$$K[\partial] \simeq \mathbb{I}_1/\int \mathbb{I}_1 \xrightarrow{*} \mathbb{I}_1/\mathbb{I}_1\partial \simeq K[\int], \quad q + \int \mathbb{I}_1 \mapsto q^* + \mathbb{I}_1\partial, \quad (48)$$

such that  $(am)^* = m^*a^*$ ,  $m^{**} = m$ ,  $(na)^* = a^*n^*$  and  $n^{**} = n$  for all elements  $a \in \mathbb{I}_1$ ,  $m \in K[\int]$  and  $n \in K[\partial]$ . Notice that  $K[\int]$  is a faithful simple left  $\mathbb{I}_1$ -module isomorphic to the faithful simple  $\mathbb{I}_1$ -module  $K[x]$ , and  $\mathbb{I}_1/\int \mathbb{I}_1$  is a faithful simple right  $\mathbb{I}_1$ -module. These are the only (up to isomorphism) faithful simple left and right modules of the algebra  $\mathbb{I}_1$  respectively.

For the right  $\mathbb{I}_1$ -module  $K[\partial]$ , let  $\text{End}_K(K[\partial])^{op}$  be its  $K$ -endomorphism algebra where we write the argument of a linear maps  $\varphi$  of  $\text{End}_K(K[\partial])^{op}$  on the *left*, i.e.,  $(\cdot)\varphi : K[\partial] \rightarrow K[\partial]$ ,  $p \mapsto (p)\varphi$ . The upper script ‘op’ indicates this fact, it also means that the algebra  $\text{End}_K(K[\partial])^{op}$  is the *opposite* algebra to the usual  $K$ -endomorphism algebra  $\text{End}_K(K[\partial])$  where we write the argument of a linear map on the *right*. Since the right  $\mathbb{I}_1$ -module  $K[\partial]$  is faithful, there is the algebra monomorphism  $\mathbb{I}_1 \rightarrow \text{End}_K(K[\partial])^{op}$ ,  $\mapsto (\cdot a : p \mapsto pa)$ . We identify the algebra  $\mathbb{I}_1$  with its isomorphic image, i.e.,  $\mathbb{I}_1 \subseteq \text{End}_K(K[\partial])^{op}$ . Let  $\mathbb{J}_1$  be the subalgebra of  $\text{End}_K(K[\partial])^{op}$  generated by the algebra  $\mathbb{I}_1$  and the set  $\mathbb{J}_1^0 := \mathbb{I}_1 \cap \text{Aut}_K(K[\partial])^{op}$  where  $\text{Aut}_K(K[\partial])^{op}$  is the group of units of the algebra  $\text{End}_K(K[\partial])^{op}$ . Using (47) and (48), we see that

$$(\mathbb{I}_1^0)^* = \mathbb{J}_1^0, \quad (\mathbb{J}_1^0)^* = \mathbb{I}_1^0. \quad (49)$$

Therefore, the maps (47) and (48) yield the  $K$ -algebra anti-isomorphisms (46).

**Corollary 8.4** *Let  $a \in \widetilde{\mathbb{I}}_1$  and  $b \in \widetilde{\mathbb{J}}_1$ . Then*

1.  $\ker(a_{K[\int]}\cdot)^* = \ker(\cdot a_{K[\partial]}^*)$  and  $\text{coker}(a_{K[\int]}\cdot)^* = \text{coker}(\cdot a_{K[\partial]}^*)$ .
2.  $\ker(\cdot b_{K[\partial]})^* = \ker(b_{K[\int]}^*\cdot)$  and  $\text{coker}(\cdot b_{K[\partial]})^* = \text{coker}(b_{K[\int]}^*\cdot)$ .

*Proof.* Straightforward, see (46).  $\square$

The next corollary is a straightforward consequence of (46) and Theorem 8.3.

**Corollary 8.5** *Let  $\mathbb{J}_1^0 := \mathbb{I}_1 \cap \text{Aut}_K(K[\partial])^{op}$  and  $\widetilde{\mathbb{J}}_1$  be the subalgebra of  $\text{End}_K(K[\partial])^{op}$  generated by the algebra  $\mathbb{I}_1$  and  $\mathbb{J}_1^{0-1}$ . Then*

1.  $F\mathbb{J}_1^{0^{-1}} = F$  but  $F \subsetneq \mathbb{J}_1^{0^{-1}}F$ .
2. The multiplicatively closed set  $\mathbb{J}_1^0$  is a left but not right Ore subset of  $\mathbb{I}_1$  and  $\mathbb{J}_1^{0^{-1}}\mathbb{I}_1 \simeq \widetilde{\mathbb{J}}_1$ .
3.  $\mathbb{J}_1^{0^{-1}}F$  is the only proper ideal of the algebra  $\widetilde{\mathbb{J}}_1$ ,  $(\mathbb{J}_1^{0^{-1}}F)^2 = \mathbb{J}_1^{0^{-1}}F$ .
4.  $\widetilde{\mathbb{J}}_1/\mathbb{J}_1^{0^{-1}}F \simeq \text{Frac}(A_1)^{op}$ .
5.  $\mathcal{C}(\widetilde{\mathbb{J}}_1) = \mathbb{J}_1^{0^{-1}}F$ .
6.  $\text{frac}_{K[\partial]}(\widetilde{\mathbb{J}}_1) \simeq \widetilde{\mathbb{J}}_1/\mathcal{C}(\widetilde{\mathbb{J}}_1) \simeq \text{Frac}(A_1)^{op}$ .
7. (a) There are only two (up to isomorphism) simple right  $\widetilde{\mathbb{J}}_1$ -modules,  $K[\partial]$  and  $\text{Frac}(A_1)^{op} = \widetilde{\mathbb{J}}_1/\mathbb{J}_1^{0^{-1}}F$ . The first one is faithful and the second one is not.  
 (b) There are only two (up to isomorphism) simple left  $\widetilde{\mathbb{J}}_1$ -modules  $\mathbb{J}_1^{0^{-1}}(\mathbb{I}_1/\mathbb{I}_1\partial) \simeq \mathbb{J}_1^{0^{-1}}K[x]$  and  $\text{Frac}(A_1)^{op}$ . The first one is faithful and the second one is not.
8.  $F_{\widetilde{\mathbb{J}}_1} = \bigoplus_{i \in \mathbb{N}} E_{i,\mathbb{N}}$  is an infinite direct sum of nonzero right ideals  $E_{i,\mathbb{N}} = e_{i0}F \simeq K[\partial]_{\widetilde{\mathbb{J}}_1}$ , therefore the algebra  $\widetilde{\mathbb{J}}_1$  is not right Noetherian.  $\mathbb{J}_1^{0^{-1}}F = \mathbb{J}_1^{0^{-1}}(\bigoplus_{i \in \mathbb{N}} E_{\mathbb{N},i}) \simeq \bigoplus_{i \in \mathbb{N}} \mathbb{J}_1^{0^{-1}}E_{\mathbb{N},i}$  is an infinite direct sum of nonzero left ideals  $\mathbb{J}_1^{0^{-1}}E_{\mathbb{N},i}$  where  $E_{\mathbb{N},i} = Fe_{0i} \simeq K[x]$ , therefore the algebra  $\widetilde{\mathbb{J}}_1$  is not left Noetherian.

## 9 The largest left and right quotient rings of the algebra $\mathbb{I}_1$

In this section, it is proved that neither left nor right quotient ring for the algebra  $\mathbb{I}_1$  exists, and the largest left and the largest right quotient rings of  $\mathbb{I}_1$  are found, they are not  $\mathbb{I}_1$ -isomorphic but  $\mathbb{I}_1$ -anti-isomorphic (Theorem 9.7). The sets of right regular, left regular and regular elements of the algebra  $\mathbb{I}_1$  are described (Lemma 9.3.(1), Corollary 9.4.(1) and Corollary 9.5).

Let  $R$  be a ring. An element  $r$  of a ring  $R$  is *right regular* if  $rs = 0$  implies  $s = 0$  for  $s \in R$ . Similarly, *left regular* is defined, and *regular* means both right and left regular (and hence not a zero divisor). We denote by  $\mathcal{C}_R$ ,  $\mathcal{C}'_R$  and  $\mathcal{C}_R'$  the sets of regular, right regular and left regular elements of  $R$  respectively. All these sets are monoids. A multiplicatively closed subset  $S$  of  $R$  is said to be a *right Ore set* if it satisfies the *right Ore condition* if, for each  $r \in R$  and  $s \in S$ ,  $rS \cap sR \neq \emptyset$ . Let  $S$  be a (non-empty) multiplicatively closed subset of  $R$ , and let  $\text{ass}(S) := \{r \in R \mid rs = 0 \text{ for some } s \in S\}$ . Then a *right quotient ring* of  $R$  with respect to  $S$  (a *right localization* of  $R$  at  $S$ ) is a ring  $Q$  together with a homomorphism  $\varphi : R \rightarrow Q$  such that

- (i) for all  $s \in S$ ,  $\varphi(s)$  is a unit of  $Q$ ,
- (ii) for all  $q \in Q$ ,  $q = \varphi(r)\varphi(s)^{-1}$  for some  $r \in R$ ,  $s \in S$ , and
- (iii)  $\ker(\varphi) = \text{ass}(S)$ .

If such a ring  $Q$  exists the ring  $Q$  is unique up to isomorphism, usually it is denoted by  $RS^{-1}$ . For a right Ore set  $S$ , the set  $\text{ass}(S)$  is an ideal of the ring  $R$ . Recall that  $RS^{-1}$  exists iff  $S$  is a right Ore set and the set  $\overline{S} = \{s + \text{ass}(S) \in R/\text{ass}(S) \mid s \in S\}$  consists of regular elements ([26], 2.1.12). Similarly, a *left Ore set*, the *left Ore condition* and the *left quotient ring*  $S^{-1}R$  are defined. If both rings  $S^{-1}R$  and  $RS^{-1}$  exist then they are isomorphic ([26], 2.1.4.(ii)). The right quotient ring of  $R$  with respect to the set  $\mathcal{C}_R$  of all regular elements is called the *right quotient ring* of  $R$ . If it exists, it is denoted by  $\text{Frac}_r(R)$  or  $Q_r(R)$ . Similarly, the *left quotient ring*,  $\text{Frac}_l(R) = Q_l(R)$ , is defined. If both left and right quotient rings of  $R$  exist then they are isomorphic and we write simply  $\text{Frac}(R)$  or  $Q(R)$  in this case. We will see that neither  $\text{Frac}_l(\mathbb{I}_1)$  nor  $\text{Frac}_r(\mathbb{I}_1)$  exists (Theorem 9.7.(1)). Therefore, we introduce the following new concepts.

*Definition.* For a ring  $R$ , a maximal with respect to inclusion right Ore set  $S$  of regular elements of  $R$  (i.e.,  $S \subseteq \mathcal{C}_R$ ) is called a *maximal regular right Ore set* in  $R$ , and the quotient ring  $RS^{-1}$  is called a *maximal right quotient ring* of  $R$ . If a maximal right Ore set  $S$  is unique we say that

$S$  is the *largest regular right Ore set* in  $R$ , and the quotient ring  $RS^{-1}$  is called the *largest right quotient ring* of  $R$  denoted  $\text{Frac}_r(R)$ . In [15], it is proved that  $\text{Frac}_r(R)$  exists for an arbitrary ring.

Notice that if  $\mathcal{C}_R$  is a right Ore set then  $\mathcal{C}_R$  is the largest regular right Ore set and the right quotient ring  $\text{Frac}_r(R)$  is the largest right quotient ring of  $R$ . That is why we use the same notation  $\text{Frac}_r(R)$  for the largest right quotient ring of  $R$ . Similarly, a *maximal regular left Ore set*, the *largest regular left Ore set* and the *largest left quotient ring* of  $R$ ,  $\text{Frac}_l(R)$ , are defined. The two natural questions below have negative solutions as the case of the algebra  $R = \mathbb{I}_1$  demonstrates (Theorem 9.7).

*Question 1.* Is the largest regular right Ore set of  $R$  also the largest regular left Ore of  $R$ ?

*Question 2.* Are the rings  $\text{Frac}_l(R)$  and  $\text{Frac}_r(R)$   $R$ -isomorphic, i.e., there is a ring isomorphism  $\varphi : \text{Frac}_l(R) \rightarrow \text{Frac}_r(R)$  such that  $\varphi(r) = r$  for all elements  $r \in R$ ?

It is obvious that if  $S$  is the largest regular left Ore set of  $R$  which is also a right Ore set and if  $S'$  is the largest regular right Ore set of  $R$  which is a left Ore set then  $S = S'$  and  $\text{Frac}_l(R) \simeq \text{Frac}_r(R)$ . In this case, we simply write  $\text{Frac}(R)$ .

The next proposition gives answers to two questions:

*Question 3.* What is the group  $\tilde{\mathbb{I}}_1^*$  of units of the algebra  $\tilde{\mathbb{I}}_1$ ?

*Question 4.* What is the image of the group  $\tilde{\mathbb{I}}_1^*$  under the natural algebra epimorphism  $\pi : \tilde{\mathbb{I}}_1 \rightarrow \tilde{\mathbb{I}}_1/\mathcal{C}(\tilde{\mathbb{I}}_1) = \text{Frac}(B_1)$ ?

Notice that the obvious concept of the degree  $\deg_\partial$  on  $B_1$  can be extended to  $\text{Frac}(B_1)$  by the rule  $\deg_\partial(s^{-1}a) = \deg_\partial(a) - \deg_\partial(s)$ .

**Proposition 9.1** 1.  $\tilde{\mathbb{I}}_1^* = \mathbb{I}_1^0 \mathbb{I}_1^{0-1} := \{ts^{-1} \mid t, s \in \mathbb{I}_1^0\}$ .

2.  $\pi(\tilde{\mathbb{I}}_1^*) = B_1^0 B_1^{0-1} \neq \text{Frac}(B_1)^*$ .

3.  $B_1^0 B_1^{0-1}$  is a normal subgroup of the group  $\text{Frac}(B_1)^*$  of units of the skew field  $\text{Frac}(B_1)$ .

4.  $\text{Frac}(B_1)^* = \bigsqcup_{i \in \mathbb{Z}} B_1^0 B_1^{0-1} \partial^i$ , a disjoint union.

5. The group  $B_1^0 B_1^{0-1}$  is the kernel of the group epimorphism  $\deg_\partial : \text{Frac}(B_1)^* = \text{Frac}(A_1)^* \rightarrow \mathbb{Z}$ .

*Proof.* 1. Let  $u \in \tilde{\mathbb{I}}_1^*$ . Then  $u = ts^{-1}$  for some elements  $t \in \mathbb{I}_1$  and  $s \in \mathbb{I}_1^0$  (Theorem 8.3.(2)). Clearly,  $t \in \mathbb{I}_1 \cap \tilde{\mathbb{I}}_1^* \subseteq \mathbb{I}_1 \cap \text{Aut}_K(K[x]) = \mathbb{I}_1^0$ .

2. The equality follows from statement 1 and (45). The inequality follows from statement 4.

4. Since  $B_1 = K[H][\partial, \partial^{-1}; \tau]$ , statement 4 is obvious.

3. Since  $B_1^0 B_1^{0-1} \partial^i = \partial^i B_1^0 B_1^{0-1}$ , statement 3 follows from statement 4.

5. Since  $\deg_\partial(B_1^0 B_1^{0-1}) = 0$  and  $\deg_\partial(\partial^i) = i$  for all  $i \in \mathbb{Z}$ , statement 5 follows from statement 4.  $\square$

Let  $S$  be a right Ore set of a ring  $R$ ,  $\overline{R} := R/\text{ass}(A)$  and  $\overline{S} = \{s + \text{ass}(S) \mid s \in S\}$ . The right Ore set  $S$  such that the elements of  $\overline{S}$  are regular in  $\overline{R}$  (and so  $RS^{-1}$  exists) is called a *right denominator set*. Similarly, a *left denominator set* is defined.

**Corollary 9.2** 1. The multiplicatively closed set  $S_\partial := \{\partial^i \mid i \in \mathbb{N}\}$  is a left denominator set in  $\tilde{\mathbb{I}}_1$  with  $\text{ass}(S_\partial) = F\mathbb{I}_1^{0-1}$  and  $S_\partial^{-1}\tilde{\mathbb{I}}_1 \simeq \text{Frac}(B_1)$ .

2. The multiplicatively closed set  $S_f := \{f^i \mid i \in \mathbb{N}\}$  is a right denominator set in  $\tilde{\mathbb{I}}_1$  with  $\text{ass}(S_f) = \mathbb{J}_1^{0-1}F$  and  $\tilde{\mathbb{I}}_1 S_f^{-1} \simeq \text{Frac}(B_1)^{op}$ .

*Proof.* 1. Since  $F = \bigcup_{i \geq 1} \ker_F(\partial^i \cdot)$  and  $F\mathbb{I}_1^{0-1}$  is the only proper ideal of the algebra  $\tilde{\mathbb{I}}_1$  (Theorem 8.3.(3)),  $\text{ass}(S_\partial) = F\mathbb{I}_1^{0-1}$  where  $\text{ass}(S_\partial) = \{a \in \mathbb{I}_1 \mid \partial^i a = 0 \text{ for some } i \geq 1\}$ .

Since  $\widetilde{\mathbb{I}}_1/\text{ass}(S_\partial) = \widetilde{\mathbb{I}}_1/F\mathbb{I}_1^{0-1} \simeq \text{Frac}(B_1)$  (Theorem 8.3.(5,6)), we have the isomorphism  $S_\partial^{-1}\widetilde{\mathbb{I}}_1 \simeq \text{Frac}(B_1)$ , by Proposition 9.1.(4).

2. Statement 2 is obtained from statement 1 by using the anti-isomorphism  $*$ , see (46).  $\square$

Lemma 9.3.(1), Corollary 9.4.(1) and Corollary 9.5 describe the sets of right regular, left regular and regular elements of the algebra  $\mathbb{I}_1$  respectively.

**Lemma 9.3** *Let  $a \in \mathbb{I}_1$ .*

1. *The following statements are equivalent.*

- (a) *The map  $a_{\mathbb{I}_1} \cdot$  is an injection.*
- (b) *The map  $a_F \cdot$  is an injection.*
- (c) *The map  $a_{K[x]} \cdot$  is an injection (see Theorem 6.6).*

2. *The following statements are equivalent.*

- (a) *The map  $a_{\mathbb{I}_1} \cdot$  is a surjection.*
- (b)  *$a = \lambda \partial^i + f$  for some  $\lambda \in K^*$ ,  $i \geq 0$  and  $f \in F$  such that the map  $a_{K[x]} \cdot$  is surjective (see Theorem 6.3).*

*Proof.* 1. If  $a \in F$  then none for the three statements is true. If  $a \notin F$  then the three statements hold since  $\ker_{\mathbb{I}_1}(a \cdot) = \ker_F(a \cdot)$  (as  $B_1 = \mathbb{I}_1/F$  is a domain) and  ${}_{\mathbb{I}_1}F \simeq K[x]^{(\mathbb{N})}$ .

2. Applying the Snake Lemma to the commutative diagram of right  $\mathbb{I}_1$ -modules

$$\begin{array}{ccccccc} 0 & \longrightarrow & F & \longrightarrow & \mathbb{I}_1 & \longrightarrow & B_1 \longrightarrow 0 \\ & & \downarrow a \cdot & & \downarrow a \cdot & & \downarrow a \cdot \\ 0 & \longrightarrow & F & \longrightarrow & \mathbb{I}_1 & \longrightarrow & B_1 \longrightarrow 0 \end{array}$$

yields the long exact sequence

$$0 \rightarrow \ker_F(a \cdot) \rightarrow \ker_{\mathbb{I}_1}(a \cdot) \rightarrow \ker_{B_1}(a \cdot) \rightarrow \text{coker}_F(a \cdot) \rightarrow \text{coker}_{\mathbb{I}_1}(a \cdot) \rightarrow \text{coker}_{B_1}(a \cdot) \rightarrow 0. \quad (50)$$

(a)  $\Rightarrow$  (b) If the map  $a_{\mathbb{I}_1} \cdot$  is a surjection then so is the map  $a_{B_1} \cdot$ , i.e.,  $\bar{a} := a + F \in B_1^*$ , and so  $\bar{a} = \lambda \partial^i$  for some  $\lambda \in K^*$  and  $i \in \mathbb{Z}$ , and then, by (50),  $\text{coker}_F(a \cdot) = \text{coker}_{\mathbb{I}_1}(a \cdot) = 0$ . Since  ${}_{\mathbb{I}_1}F \simeq K[x]^{(\mathbb{N})}$ , the map  $a_{K[x]} \cdot$  is a surjection. By Theorem 6.3,  $i \geq 0$ .

(a)  $\Leftarrow$  (b) If statement (b) holds then  $a + F \in B_1^*$  and  $\text{coker}_F(a \cdot) = 0$  (since  ${}_{\mathbb{I}_1}F \simeq K[x]^{(\mathbb{N})}$ ), then, by (50),  $0 = \text{coker}_F(a \cdot) = \text{coker}_{\mathbb{I}_1}(a \cdot)$ .  $\square$

It is obvious that  $a_{\mathbb{I}_1} \cdot$  is an injection (resp. a surjection) iff  $\cdot a_{\mathbb{I}_1}^*$  is an injection (resp. a surjection). Using the involution  $*$  defined in (46),  $F^* = F$ ,  ${}_{\mathbb{I}_1}K[x] \simeq \mathbb{I}_1/\mathbb{I}_1\partial \simeq K[f]$  and  $(\mathbb{I}_1/\mathbb{I}_1\partial)^* = \mathbb{I}_1/\int \mathbb{I}_1 \simeq K[\partial]_{\mathbb{I}_1}$ , we obtain similar results but for right multiplication.

**Lemma 9.4** *Let  $a \in \mathbb{I}_1$ .*

1. *The following statements are equivalent.*

- (a) *The map  $\cdot a_{\mathbb{I}_1}$  is an injection.*
- (b) *The map  $\cdot a_F$  is an injection.*
- (c) *The map  $\cdot a_{K[\partial]} \cdot$  is an injection.*

2. *The following statements are equivalent.*

- (a) *The map  $\cdot a_{\mathbb{I}_1}$  is a surjection.*

- (b)  $a = \lambda \int^i + f$  for some  $\lambda \in K^*$ ,  $i \geq 0$  and  $f \in F$  such that the map  $\cdot a_{K[\partial]}$  is surjective (see Theorem 6.6).

**Corollary 9.5** *Let  $a \in \mathbb{I}_1$ . Then  $a \in \mathcal{C}_{\mathbb{I}_1}$  iff the maps  $a_{K[x]}$  and  $a_{K[x]}^*$  are injections (Theorem 6.6).*

The next lemma is a useful criterion of left regularity.

**Lemma 9.6** *Let  $a \in \mathbb{I}_1 \setminus F$ . Then the map  $\cdot a_{\mathbb{I}_1}$  is an injection iff for all natural numbers  $n \geq 0$ ,  $(e_{00} + e_{11} + \cdots + e_{nn})\text{im}(a_{K[x]}) = K[x]_{\leq n}$ . For example, the  $\cdot(\partial + \int)_{\mathbb{I}_1}$  is an injection.*

*Proof.* The map  $\cdot a_{\mathbb{I}_1}$  is not an injection iff there exists a nonzero element  $f \in F$  such that  $fa = 0$  since  $a \notin F$  and  $\mathbb{I}_1/F$  is a domain iff  $(e_{00} + e_{11} + \cdots + e_{nn})\text{im}(a_{K[x]}) \neq K[x]_{\leq n}$  for some natural number  $n$  (e.g.  $n = \deg_F(f)$  since  $f = f(e_{00} + e_{11} + \cdots + e_{nn})$ ): if  $V := (e_{00} + e_{11} + \cdots + e_{nn})\text{im}(a_{K[x]}) \neq K[x]_{\leq n}$  then  $K[x]_{\leq n} = V \oplus V'$  for some nonzero subspace  $V'$  of  $K[x]_{\leq n}$  and take  $f$  to be the projection onto  $V'$ .

Let  $a = \partial + \int$  then since  $a * x^{[0]} = x^{[1]}$ ,  $a * x^{[i]} = x^{[i-1]} + x^{[i+1]}$  for  $i \geq 1$ , we have  $(e_{00} + e_{11} + \cdots + e_{nn})\text{im}(a_{K[x]}) = K[x]_{\leq n}$  for all  $n \geq 0$ . Then, the map  $\cdot a_{\mathbb{I}_1}$  is an injection, by Lemma 9.6.  $\square$

Let  $R$ ,  $S$  and  $T$  be rings such that  $R \subseteq S$  and  $R \subseteq T$ . If there exists a ring isomorphism (respectively, an anti-isomorphism)  $\varphi : S \rightarrow T$  such that  $\varphi(r) = r$  for all  $r \in R$  (respectively,  $\varphi(R) = R$ ) we say that the rings  $S$  and  $T$  are  $R$ -isomorphic (respectively,  $R$ -anti-isomorphic).

**Theorem 9.7** 1. *The set  $\mathcal{C}_{\mathbb{I}_1}$  of regular elements of the algebra  $\mathbb{I}_1$  satisfies neither left nor right Ore condition. Therefore, the left and the right quotient rings of  $\mathbb{I}_1$ ,  $\mathcal{C}_{\mathbb{I}_1}^{-1}\mathbb{I}_1$  and  $\mathbb{I}_1\mathcal{C}_{\mathbb{I}_1}^{-1}$ , do not exist.*

2. *The set  $\mathbb{I}_1^0$  is the largest regular right Ore set in  $\mathbb{I}_1$ , and so the largest right quotient ring of fractions  $\text{Frac}_r(\mathbb{I}_1) := \mathbb{I}_1\mathbb{I}_1^0{}^{-1} = \widetilde{\mathbb{I}}_1$  of  $\mathbb{I}_1$  exists.*
3. *The set  $\mathbb{J}_1^0 = (\mathbb{I}_1^0)^*$  is the largest regular left Ore set in  $\mathbb{I}_1$ , and so the largest left quotient ring of fractions  $\text{Frac}_l(\mathbb{I}_1) := \mathbb{J}_1^0{}^{-1}\mathbb{I}_1 = \widetilde{\mathbb{J}}_1$  of  $\mathbb{I}_1$  exists.*
4. *The rings  $\text{Frac}_r(\mathbb{I}_1)$  and  $\text{Frac}_l(\mathbb{I}_1)$  are not  $\mathbb{I}_1$ -isomorphic but are  $\mathbb{I}_1$ -anti-isomorphic (see (46)). In particular, the largest regular right Ore set in  $\mathbb{I}_1$  is not a left Ore set and the largest regular left Ore set in  $\mathbb{I}_1$  is not a right Ore set.*

*Proof.* 1. By Lemma 9.6, the element  $a := \partial + \int \in \mathbb{I}_1$  is left regular, hence it is regular since  $a^* = a$ . To prove statement 1 it suffices to show that  $\mathbb{I}_1 e_{00} \cap \mathbb{I}_1 a = 0$  and  $e_{00}\mathbb{I}_1 \cap a\mathbb{I}_1 = 0$ . In fact, it suffices to show that the first equality holds since then the second equality can be obtained from the first by applying the involution  $*$ . Suppose that the first equality fails, then we can find a nonzero element, say  $u$ , in the intersection, we seek a contradiction. Since  $\mathbb{I}_1 e_{00} = \bigoplus_{i \in \mathbb{N}} K e_{i0}$ , the element  $u$  is the unique sum  $\sum_{i \in I} \lambda_i e_{i0}$  where all  $\lambda_i \in K^*$  and  $I$  is a non-empty finite subset of natural numbers. Clearly,  $u = ga$  for some element  $g \in \mathbb{I}_1$ . Since  $u \in F$  and  $a \notin F$ , we must have  $g \in F$ . Since  $F_{\mathbb{I}_1} = \bigoplus_{i \in \mathbb{N}} E_{i,\mathbb{N}}$  is the direct sum of right submodules  $E_{i,\mathbb{N}} = \bigoplus_{j \in \mathbb{N}} K e_{ij} \simeq K[\partial]_{\mathbb{I}_1}$  and  $e_{i0} \in E_{i,\mathbb{N}}$  for all  $i \in I$ , we see that  $e_{i0} = \lambda_i^{-1} e_{iig} a$ , i.e., the element  $1 \in K[\partial]$  belongs to  $\text{im}(\cdot a_{K[\partial]})$  which is obviously impossible since  $\partial^i * a = \begin{cases} \partial & \text{if } i = 0, \\ \partial^{i+1} + \partial^{i-1} & \text{if } i > 0. \end{cases}$

2. Suppose that  $S$  is a regular right Ore set in  $\mathbb{I}_1$ . We have to show that  $S \subseteq \mathbb{I}_1^0$ . Let  $u \in S$ , we have to show that  $u \in \mathbb{I}_1^0$ . Recall that  $\widetilde{\mathbb{I}}_1 = \mathbb{I}_1\mathbb{I}_1^0{}^{-1}$ ,  $F\mathbb{I}_1^0{}^{-1}$  is the only proper ideal of the algebra  $\widetilde{\mathbb{I}}_1$  and  $\widetilde{\mathbb{I}}_1/F\mathbb{I}_1^0{}^{-1} \simeq \text{Frac}(B_1)$  (Theorem 8.3). Let  $\pi : \widetilde{\mathbb{I}}_1 \rightarrow \widetilde{\mathbb{I}}_1/F\mathbb{I}_1^0{}^{-1}$ ,  $a \mapsto \bar{a} := a + F\mathbb{I}_1^0{}^{-1}$ . None of the elements of the ideal  $F\mathbb{I}_1^0{}^{-1}$  is regular. Therefore,  $u \notin F\mathbb{I}_1^0{}^{-1}$ . Then  $\pi(u) \in \text{Frac}(B_1)^*$ , and so  $u = \partial^i s t^{-1} + f t_1^{-1}$ ,  $\int^j s t^{-1} + f t_1^{-1}$  for some  $i \geq 0$ ,  $j > 0$ ,  $s, t, t_1 \in \mathbb{I}_1^0$  and  $f \in F$  (Proposition 9.1).

*Case 1:*  $u = \partial^i s t^{-1} + f t_1^{-1}$ . Taking a common right denominator we may assume that  $t = t_1$  (changing  $s$  and  $f$  if necessary), i.e.,  $u = (\partial^i s + f) t^{-1}$ . Since  $\partial^i F = F$ , we may assume that

$u = \partial^i(s + g)t^{-1}$  for some element  $g \in F$  such that  $\partial^i g = f$ . Notice that  $u \in \text{End}_K(K[x])$ , and (using Lemma 3.5).

$$\text{ind}_{K[x]}(u) = \text{ind}_{K[x]}(\partial^i) + \text{ind}_{K[x]}(s + g) + \text{ind}_{K[x]}(t^{-1}) = i + \text{ind}_{K[x]}(s) = i \geq 0.$$

The element  $u$  is regular and  ${}_{\mathbb{I}_1}F \simeq K[x]^{(\mathbb{N})}$ , hence  $i = 0$ , i.e.,  $u = (s + g)t^{-1}$  and  $\ker_{K[x]}(u) = 0$ . The two conditions  $\text{ind}_{K[x]}(u) = 0$  and  $\ker_{K[x]}(u) = 0$  imply that the map  $u_{K[x]}$  is a bijection, and so  $u \in \mathbb{I}_1 \cap \text{Aut}_K(K[x]) = \mathbb{I}_1^0$ .

*Case 2:*  $u = \int^j st^{-1} + ft_1^{-1}$ ,  $j > 0$ . By the same reason as in Case 1, we may assume that  $t = t_1$ , i.e.,  $u = (\int^j s + f)t^{-1}$ . Then (using Lemma 3.5)

$$\text{ind}_{K[x]}(u) = \text{ind}_{K[x]}(\int^j s + f) + \text{ind}_{K[x]}(t^{-1}) = \text{ind}_{K[x]}(\int^j s) = \text{ind}_{K[x]}(\int^j) + \text{ind}_{K[x]}(s) = -j < 0.$$

Therefore, we can fix an element, say  $v \in K[x]$ , such that  $v \notin \text{im}_{K[x]}(u)$ . Fix an element, say  $e \in F$ , such that  $\text{im}_{K[x]}(e \cdot) = Kv$ . Then  $e\mathbb{I}_1 \cap u\mathbb{I}_1 = 0$  since, otherwise, we get a contradiction: fix  $0 \neq w \in e\mathbb{I}_1 \cap u\mathbb{I}_1$ , then  $w = ea = ub$  for some elements  $a, b \in \mathbb{I}_1$ , and so  $0 \neq wK[x] \subseteq \text{im}_{K[x]}(e \cdot) \cap \text{im}_{K[x]}(u \cdot) = Kv \cap \text{im}_{K[x]}(u \cdot) = 0$ , a contradiction. The equality  $e\mathbb{I}_1 \cap u\mathbb{I}_1 = 0$  implies  $eS \cap u\mathbb{I}_1 = \emptyset$ , this contradicts to the fact that  $S$  is a right Ore set in  $\mathbb{I}_1$ . This means that  $S \subseteq \mathbb{I}_1^0$ , as required.

3. Statement 3 follows from statement 2 using (46).

4. Suppose that  $\varphi : \mathbb{I}_1\mathbb{I}_1^{0-1} \rightarrow \mathbb{J}_1^{0-1}\mathbb{I}_1$  is an  $\mathbb{I}_1$ -isomorphism, we seek a contradiction. By Proposition 9.8,  $\mathbb{J}_1^0 \subseteq (\mathbb{I}_1\mathbb{I}_1^{0-1})^* = \mathbb{I}_1^0\mathbb{I}_1^{0-1}$ , by Proposition 9.1.(1). Clearly,  $1 + \partial \in \mathbb{I}_1^0$ , then  $u := 1 + \int = (1 + \partial)^* \in (\mathbb{I}_1^0)^* = \mathbb{J}_1^0$ , by (49). By Proposition 6.1.(1),  $\text{ind}_{K[x]}(1 + \int) = -1$ . Since  $u \in \mathbb{I}_1\mathbb{I}_1^{0-1} \subseteq \text{Aut}_K(K[x])$ , we must have  $\text{ind}_{K[x]}(1 + \int) = 0$ , a contradiction.  $\square$

**Proposition 9.8** *Let  $R$  be a ring,  $T$  and  $S$  be a left and right denominator set of regular elements of the ring  $R$  respectively (and so  $R \subseteq T^{-1}R$  and  $R \subseteq RS^{-1}$ ). Then there is an  $R$ -isomorphism of rings  $\varphi : T^{-1}R \rightarrow RS^{-1}$  (i.e.,  $\varphi(r) = r$  for all  $r \in R$ ) iff  $S \subseteq (T^{-1}R)^*$  and  $T \subseteq (RS^{-1})^*$  where  $(T^{-1}R)^*$  and  $(RS^{-1})^*$  are the groups of units of the rings  $T^{-1}R$  and  $RS^{-1}$  respectively.*

*Proof.* ( $\Rightarrow$ ) Trivial.

( $\Leftarrow$ ) The subring  $\mathcal{T}$  of  $RS^{-1}$  generated by  $R$  and the set  $T^{-1} = \{t^{-1} \mid t \in T\}$  is canonically isomorphic to the ring  $T^{-1}R$  since  $T$  is a left denominator set of regular elements of the ring  $R$ . Similarly, the subring  $\mathcal{C}$  of  $T^{-1}R$  generated by  $R$  and the set  $S^{-1} = \{s^{-1} \mid s \in S\}$  is canonically isomorphic to the ring  $RS^{-1}$ . Therefore, we have the ring  $R$ -monomorphisms  $\varphi : T^{-1}R \rightarrow RS^{-1}$  and  $\psi : RS^{-1} \rightarrow T^{-1}R$ . Since  $\psi\varphi(T^{-1}R) = T^{-1}R$  and  $\varphi\psi(RS^{-1}) = RS^{-1}$ , the maps  $\varphi$  and  $\psi$  are surjective, and so  $\varphi$  is an  $R$ -isomorphism.  $\square$

**Corollary 9.9** *The inclusion  $A_1 \rightarrow \mathbb{I}_1$  cannot be lifted neither to a ring monomorphism  $\text{Frac}(A_1) \rightarrow \text{Frac}_l(\mathbb{I}_1)$  nor to  $\text{Frac}(A_1) \rightarrow \text{Frac}_r(\mathbb{I}_1)$ .*

*Proof.* The element  $\partial$  of the Weyl algebra  $A_1$  is invertible in  $\text{Frac}(A_1)$  but is not in  $\text{Frac}_l(\mathbb{I}_1)$  and  $\text{Frac}_r(\mathbb{I}_1)$  since  $\partial e_{00} = 0$ .  $\square$

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Department of Pure Mathematics  
University of Sheffield  
Hicks Building  
Sheffield S3 7RH  
UK  
email: v.bavula@sheffield.ac.uk